Week 1: Review of MATH 1030 (textbook Ch. 1-4)

Vector Spaces

- In this course, $\mathbb{F}$ will denote a field (see Appendix C) which you can take as $\mathbb{R}$ or $\mathbb{C}$. Very rarely we would consider other fields like $\mathbb{Q}$ or $\mathbb{Z}_2$.

- A vector space over $\mathbb{F}$ consists of a triple:

$$(\underbrace{V}_{\text{set of vectors}}, \underbrace{+}_{\text{addition}}, \underbrace{\cdot}_{\text{scalar multiplication}})$$

Satisfying the following properties:

(VS 1): $x+y = y+x \quad \forall x,y \in V$

(VS 2): $(x+y)+z = x+(y+z) \quad \forall x,y,z \in V$

(VS 3): $\exists\, 0 \in V$ s.t. $x+0 = x \quad \forall x \in V$

(VS 4): $\forall x \in V, \exists\, y \in V$ s.t. $x+y = 0$

(VS 1)–(VS 4): $(V,+)$ forms an "abelian group"

(VS 5): $1 \cdot x = x \quad \forall x \in V$

(VS 6): $(ab)\cdot x = a\cdot(b\cdot x) \quad \forall a,b \in \mathbb{F}, \forall x \in V$

(VS 7): $a\cdot(x+y) = a\cdot x + a\cdot y \quad \forall a \in \mathbb{F}, \forall x,y \in V$

(VS 8): $(a+b)\cdot x = a\cdot x + b\cdot x \quad \forall a,b \in \mathbb{F}, \forall x \in V$

- Examples of vector spaces:

(a) $\mathbb{F}^n$: n-tuples

(b) $M_{m\times n}(\mathbb{F})$: $m \times n$ matrices

(c) $C(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{R} \text{ continuous} \}$

(d) $\{\text{polynomials over } \mathbb{F}\} = P(\mathbb{F})$

(c), (d): infinite dimensional!

- A subspace is a subset $W \subseteq V$ s.t.

  (a) $0 \in W$

  (b) $x, y \in W \Rightarrow x+y \in W$ (closed under addition)

  (c) $x \in W,\ c \in \mathbb{F} \Rightarrow c \cdot x \in W$ (closed under scalar multi.)

- Examples of subspaces:

  (a) $\{\text{symmetric matrices}\} \subseteq M_{n\times n}(\mathbb{F})$

  (b) $\{0\},\ V$ - trivial subspaces

  (c) $P_n(\mathbb{F}) := \{\text{polynomials of deg} \leq n\} \subseteq P(\mathbb{F})$

  (d) $C^k(\mathbb{R}) \subseteq C(\mathbb{R})$

  (e) $\{M \text{ trace-free, i.e. } tr(M) = 0\} \subseteq M_{n\times n}(\mathbb{F})$

Thm: W_1, W_2 subspaces $\Rightarrow W_1 \cap W_2$ subspace

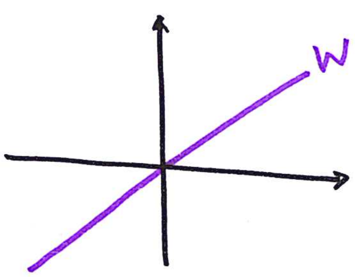
Thm: W_1, W_2 subspaces $\Rightarrow W_1 + W_2$ subspace

$$W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1,\ w_2 \in W_2\}$$

Caution: W_1, W_2 subspace $\not\Rightarrow W_1 \cup W_2$ subspace !

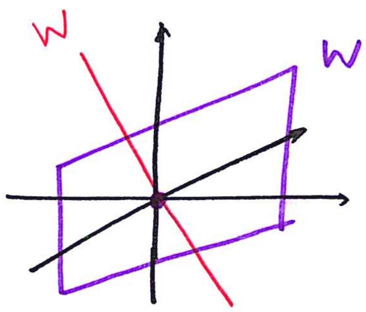
- A subspace $W \subseteq V$ is itself a vector space.

(nontrivial) subspaces of $\mathbb{R}^2$



lines through 0.

(nontrivial) subspaces of $\mathbb{R}^3$



lines / planes through 0.

- For any subset $S \subseteq V$, the span of S is the smallest subspace containing S, i.e.

$$\text{span}(S) = \{ \underbrace{\sum_{i=1}^{k} c_i \cdot w_i}_{\text{linear combination of vectors of } S} : c_i \in \mathbb{F},\ w_i \in S \}$$

- A subset $S \subseteq V$ is linearly dependent if $\exists\ a_1, \dots, a_k \in \mathbb{F}$ (not all 0) and distinct $w_i \in S$ s.t. $a_1 w_1 + a_2 w_2 + \cdots + a_k w_k = \vec{0}$.

Otherwise, S is linearly independent.

Thm: S linearly independent iff the following holds:

" $a_1 w_1 + \cdots + a_k w_k = 0 \Rightarrow a_1 = \cdots = a_k = 0$.
$a_i \in \mathbb{F},\ w_i \in S$ "

Thm: (a) $\underset{\text{lin. dep.}}{S_1} \subseteq S_2 \Rightarrow S_2$ lin. dep.

(b) $S_1 \subseteq \underset{\text{lin. indep.}}{S_2} \Rightarrow S_1$ lin. indep.

Thm: Let $S \subseteq V$ be a linearly indep. subset.

$$S \cup \{v\} \text{ lin. dep.} \iff v \in \text{span}(S).$$

- A subset $\beta \subseteq V$ is a basis for V if

(a) β is lin. indep.

(b) $\text{span}(\beta) = V$.

Thm: β is a basis for $V \iff$ for each $v \in V$, $\exists$ unique $c_i \in \mathbb{F}$, $w_i \in \beta$ st

$$v = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

- **Examples of basis:**

(a) $\mathbb{F}^n$: $\beta = \{e_1, \dots, e_n\}$ $e_i = (0, \dots, 0, \overset{i^{th}\text{ entry}}{\overset{\downarrow}{1}}, 0, \dots, 0)$

(b) $P_n(\mathbb{F})$: $\beta = \{1, x, x^2, \dots, x^n\}$

(c) $P(\mathbb{F})$: $\beta = \{1, x, x^2, \dots \dots\}$ infinite!

- A vector space V over $\mathbb{F}$ is finite dimensional if $\exists$ finite basis β.

$\dim(V)$ = number of elements in β
(indep of the choice of β)

Otherwise, V is infinite dimensional.

Thm: Let V be an n-dimensional vector space over $\mathbb{F}$.

(a) S lin. indep. subset $\Rightarrow$ # of elements in $S \le n$.

(b) $\mathrm{span}(S) = V$ $\Rightarrow$ # of elements of $S \ge n$.

(c) Any lin. indep. subset $S \subseteq V$ can be extended to a basis for V.

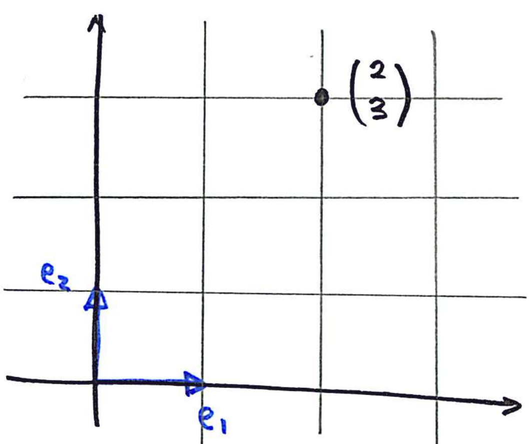
(d) $W \subseteq V$ subspace $\Rightarrow$ $\dim W \le \dim V$

Moreover, "=" holds iff $W = V$.

FACT: Any n-dim. vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$, but not "canonically". For example, once we fix a basis β

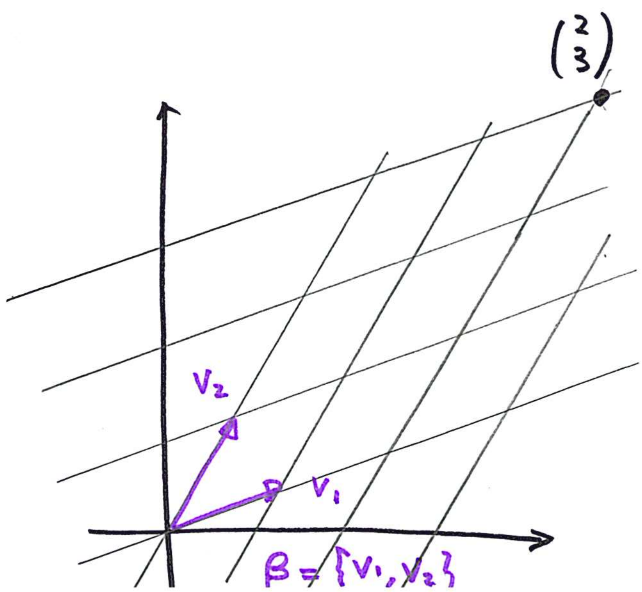
$\beta = \{v_1, \dots, v_n\}$ $V \cong \mathbb{F}^n$ (BUT depends on β)

$$v = a_1 v_1 + \dots + a_n v_n \longleftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$



$\beta = \{e_1, e_2\}$ standard basis

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ means different things in different "coordinate system"



System of Linear Equations

- Given a system of linear equations: (unknowns: $x_1, \dots, x_n$)

$$(\star)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

there are 3 possibilities:

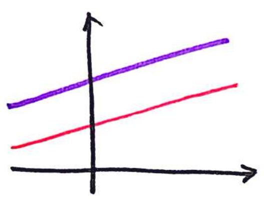
(I) No solution

(II) Exactly 1 solution

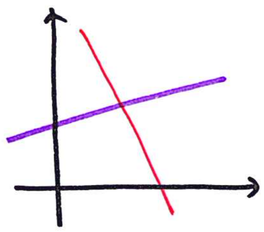
(III) Infinitely many solution

- To find solution(s), do Gaussian elimination (see textbook 3.4)
- Geometrically, say when $m = n = 2$,

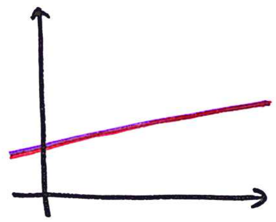
(I) No solution



(II) exactly 1 solution



(III) Infinitely many solution



Matrices

- We can write $(\star)$ in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\boxed{Ax = b}$$

- Algebraic operations for matrices:

$$\underbrace{A \pm B, \quad cA,}_{M_{m\times n}(\mathbb{F}) \text{ forms a vector space.}} \qquad \underset{m\times n \quad n\times k}{AB} = m\times k \text{ matrix.}$$

Non-commutativity: $\boxed{AB \neq BA}$ in general.

- Special matrices:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

zero matrix

identity matrix: $\boxed{AI = A = IA}$ for all A. (square matrix)

- A square matrix A is invertible if there exists a matrix A^{-1} called the inverse of A s.t. $\boxed{AA^{-1} = I = A^{-1}A.}$

- Finding inverse: $(A \mid I) \xrightarrow[\text{operations}]{\text{row}} (I \mid A^{-1})$

Thm: A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$

- Computing $\det(A)$:

2×2: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} := ad - bc$

3×3: $$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} := a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$n\times n$: inductively defined. (actually we can expand along any rows/columns, just remember alternating signs)

- Important properties: $\boxed{\det(A) = \det(A^t)}$ & $\boxed{\det(AB) = \det(A)\det(B)}$ "multiplicative"

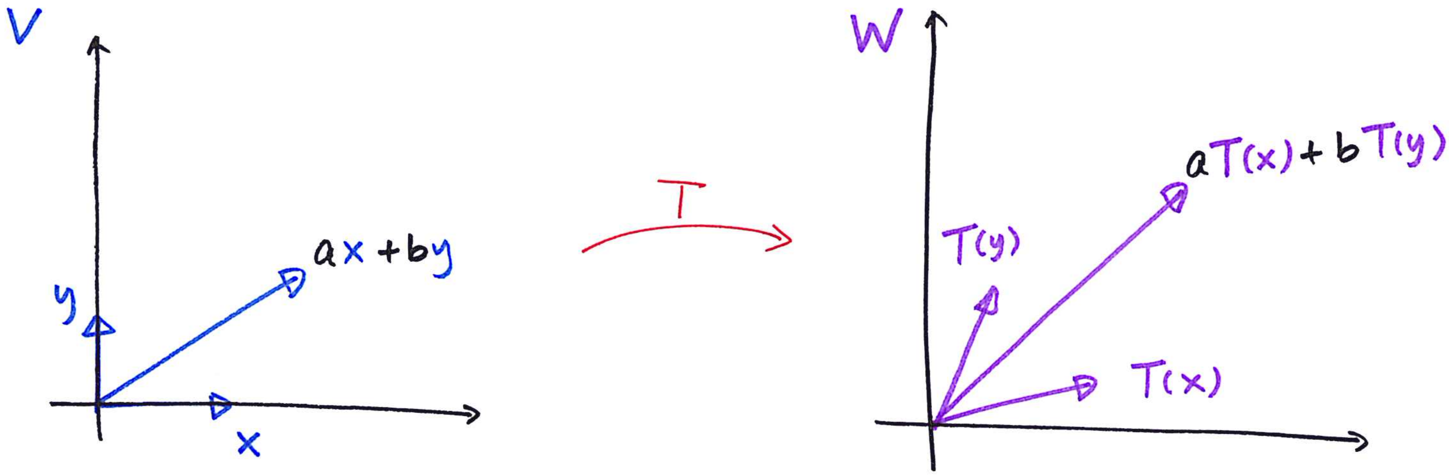
Linear Transformation

V, W : vector spaces over $\mathbb{F}$

- A map $T : V \longrightarrow W$ is linear if $\forall a, b \in \mathbb{F}$, $\forall x, y \in V$

$$T(ax + by) = aT(x) + bT(y)$$

i.e. T respects the vector space structure of V and W.



- Examples of linear transformations

(a) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

rotation : $T(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta,\ a_1 \sin\theta + a_2 \cos\theta)$.

reflection : $T(a_1, a_2) = (a_1, -a_2)$.

projection : $T(a_1, a_2) = (a_1, 0)$.

(b) $T : M_{m\times n}(\mathbb{F}) \longrightarrow M_{n\times m}(\mathbb{F})$, $T(A) = A^t$

(c) $T : P_n(\mathbb{R}) \longrightarrow P_{n-1}(\mathbb{R})$, $T(f(x)) = f'(x)$

(d) $T : C(\mathbb{R}) \longrightarrow \mathbb{R}$, $T(f) = \int_a^b f(t)\,dt$, $a<b$ fixed.

(e) Identity Transformation: $I_V : V \to V$, $\boxed{I_V(x) = x}$ $\forall x \in V$

(f) Zero Transformation: $T_0 : V \longrightarrow W$, $\boxed{T_0(x) = 0}$ $\forall x \in V$

- Given a linear map $T : V \longrightarrow W$, define two <u>subspaces</u>:

<u>kernel / null space</u>: $N(T) := \{ x \in V \mid T(x) = 0 \} \subseteq V$

<u>range / image</u>: $R(T) := \{ T(x) \mid x \in V \} \subseteq W$

Dimension Theorem: $\boxed{\underbrace{\dim N(T)}_{\text{"nullity"}} + \underbrace{\dim R(T)}_{\text{"rank"}} = \dim V}$

<u>Thm</u>: (i) T is one-to-one $\iff N(T) = \{0\}$

(ii) T is onto $\iff R(T) = W$

[If both (i) (ii) hold, T is an isomorphism.]

Matrices & Linear Transformations

- Given $A \in M_{m\times n}(\mathbb{F})$, one can define a linear transformation

$$T : \mathbb{F}^n \longrightarrow \mathbb{F}^m, \qquad T(x) := \underbrace{\overset{m\times n}{A}\ \overset{n\times 1}{x}}_{\text{matrix multiplication}}$$

we also write T as L_A. (left multiplication)

<u>Examples</u>: $A = O \implies L_A =$ zero transformation T_0

$A = I \implies L_A =$ identity transformation $I_{\mathbb{F}^n}$ $(n = m)$

- In fact, <u>ANY</u> linear transformation $T : V \to W$ can be expressed in this form <u>AFTER</u> we pick an ordered basis β_V and β_W for V and W respectively:

$\dim V = n$
$\dim W = m$

$$\left\{ \begin{array}{c} \text{linear transformation} \\ T : V \to W \end{array} \right\} \xleftrightarrow{\ \cong\ } \left\{ A \in M_{m\times n}(\mathbb{F}) \right\}$$

but not "canonical"

depends on the choice of bases β_V & β_W

$m\times n$ matrices

- Recall that if $\beta = \{u_1, u_2, \dots, u_n\}$ is an <u>ordered</u> basis for an n-dim. vector space V, then any $x \in V$ can be written <u>uniquely</u> as: $x = a_1 u_1 + \cdots + a_n u_n$, $a_i \in \mathbb{F}$.

Thus, we have an isomorphism (depends on β!):

$$\begin{tikzcd} V \arrow[r, leftrightarrow, "\cong_\beta"] & \mathbb{F}^n \\ x \arrow[u, phantom, "\in" description] \arrow[r, leftrightarrow] & {[x]_\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}} \arrow[u, phantom, "\in" description] \end{tikzcd}$$

- Given ordered bases:

$\beta = \{v_1, v_2, \dots, v_n\}$ for V

$\gamma = \{w_1, w_2, \dots, w_m\}$ for W

and a linear map $T: V \longrightarrow W$, we have the diagram:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, leftrightarrow, "\cong_\beta"'] & W \arrow[d, leftrightarrow, "\cong_\gamma"] \\ \mathbb{F}^n \arrow[r, "L_A"] & \mathbb{F}^m \end{tikzcd}$$

$$\boxed{A = [T]_\beta^\gamma}$$

Matrix representation of T in the ordered bases β and γ

<u>Examples</u>: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\beta = \{e_1, e_2\} = \gamma$ standard basis

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

T = rotation by θ (counterclockwise)

T = reflection (about x-axis)

T = projection (on the x-axis)

Denote $\mathcal{L}(V,W) = \{ T: V \to W \text{ linear} \}$

$M_{m\times n}(\mathbb{F}) = \{ A: m\times n \text{ matrices (coeff. in } \mathbb{F}) \}$

which are vector spaces over $\mathbb{F}$. Here, $m = \dim W$, $n = \dim V$.

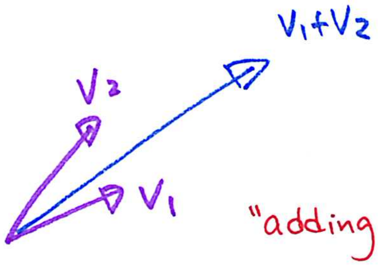
We have the following dictionary: (depends on choice of β, γ)

$\mathcal{L}(V,W)$ $\quad [T]_\beta^\gamma = A$	$M_{m\times n}(\mathbb{F})$
linear transformation $T: V \to W$	matrix A
sum $T + U$	sum $A + B$
scalar multiplication $c \cdot T$	scalar multiplication $c \cdot A$
composition $T \circ U$	matrix multiplication AB
inverse T^{-1}	inverse A^{-1}

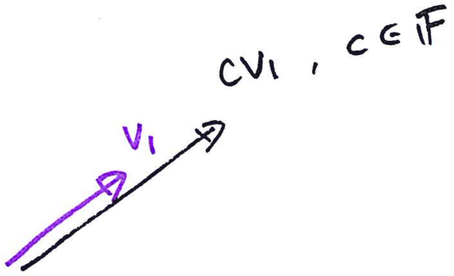
different languages to represent the same concept.

Supplementary Notes:

①

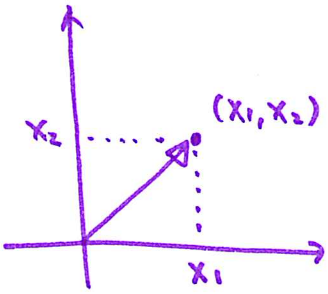


"adding vectors"



"rescaling vectors"

② $\mathbb{R}^2 := \{(x_1, x_2) : x_i \in \mathbb{R}\}$.



$\mathbb{R}^2$ = plane

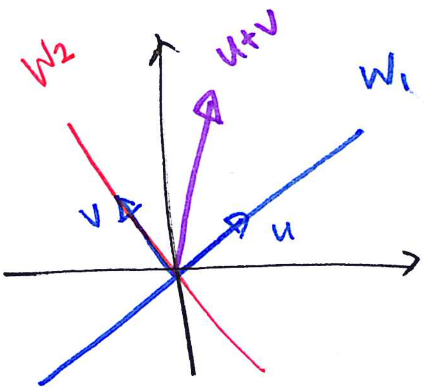
③ $M_{2\times 3}(\mathbb{R})$: as a vector space (over $\mathbb{R}$)

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$2\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

④ Symmetric matrix: 2×2 $\quad A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$

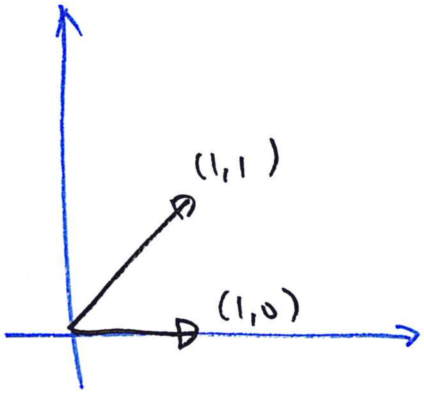
⑤ W_1, W_2 subspaces $\not\Rightarrow$ $W_1 \cup W_2$ subspace.



Note: $u, v \in W_1 \cup W_2$

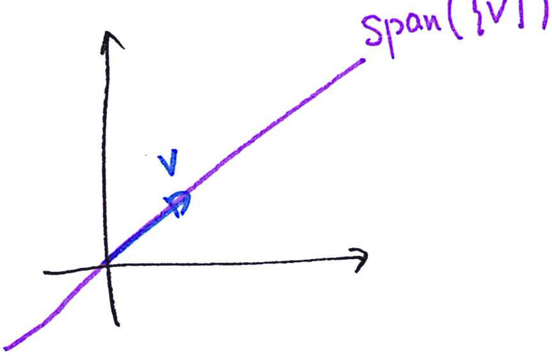
but $u+v \notin W_1 \cup W_2$.

⑥ $S = \{(1,0),(1,1)\} \subset \mathbb{R}^2$



$\text{span}(S) = \mathbb{R}^2$

"two linearly indep. vectors in $\mathbb{R}^2$ span $\mathbb{R}^2$"



"The span of a single vector v is a "line" through v and the origin"

⑦ $C(\mathbb{R})$ vector space.

Q: $\sin x \perp \cos x$? need additional "structure" to measure angles between vectors

⑧

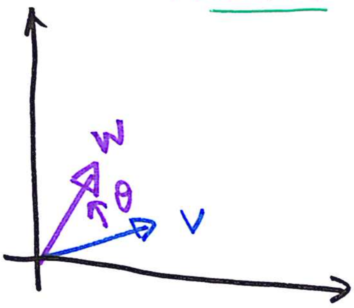
$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 & 0 \\ 0 & 2\cdot 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$$

$AB = BA$ if A & B are diagonal matrices

Pictures

rotation

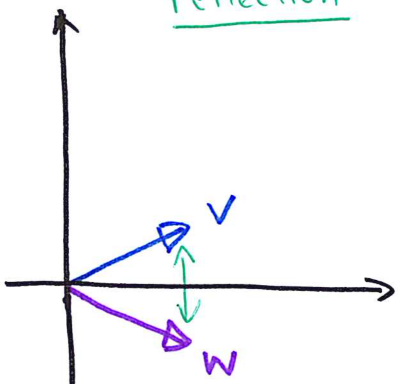


$N(T) = \{0\}$

$R(T) = \mathbb{R}^2$

"$0 + 2 = 2$"

reflection

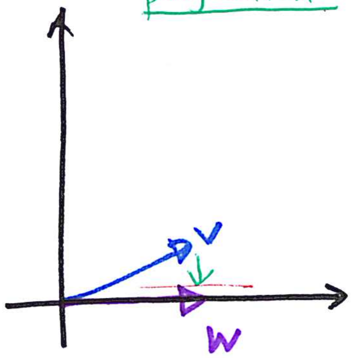


$N(T) = \{0\}$

$R(T) = \mathbb{R}^2$

"$0 + 2 = 2$"

projection



$N(T) = \{y\text{-axis}\}$

$R(T) = \{x\text{-axis}\}$

"$1 + 1 = 2$"

For I_V, $N(I_V) = \{0\}$, $R(I_V) = V$

For T_0, $N(T_0) = V$, $R(T_0) = \{0\}$