MATH 5061 Riemannian Geometry

Solution to Problem Set 4

Problem 1

We use the normal coordinate to compute the second Bianchi Identity. Choose $p \in M$ with the normal coordinate $e_1, \cdots, e_n$ at p. So we have $\nabla_{e_i} e_j = 0$ at p for any $1 \leq i, j \leq n$ and hence $[e_i, e_j] = 0$ at p.

So at p, the coderivative of Riemann curvature tensor can be written as

$$\begin{aligned}(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) &= \frac{\partial}{\partial x_i} R(e_j, e_k, e_l, e_m) \\ &= -\left\langle \nabla_{e_i} \nabla_{e_j} \nabla_{e_k} e_l, e_m \right\rangle + \left\langle \nabla_{e_i} \nabla_{e_k} \nabla_{e_j} e_\lambda e_m \right\rangle\end{aligned}$$

So

$$\begin{aligned}&(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) + (\nabla_{e_j} R)(e_k, e_i, e_l, e_m) + (\nabla_{e_k} R)(e_i, e_j, e_l, e_m) \\ = &-\boxed{\left\langle \nabla_{e_i} \nabla_{e_j} \nabla_{e_k} e_l, e_m \right\rangle} + \underline{\left\langle \nabla_{e_i} \nabla_{e_k} \nabla_{e_j} e_l, e_m \right\rangle} \\ &- \underline{\underline{\left\langle \nabla_{e_j} \nabla_{e_k} \nabla_{e_i} e_l, e_m \right\rangle}} + \boxed{\left\langle \nabla_{e_j} \nabla_{e_i} \nabla_{e_k} e_l, e_m \right\rangle} \\ &- \underline{\left\langle \nabla_{e_k} \nabla_{e_i} \nabla_{e_j} e_l, e_m \right\rangle} + \underline{\underline{\left\langle \nabla_{e_k} \nabla_{e_j} \nabla_{e_i} e_l, e_m \right\rangle}} \\ = &R(e_i, e_j, \nabla_{e_k} e_l, e_m) + R(e_j, e_k, \nabla_{e_i} e_l, e_m) + R(e_k, e_i, \nabla_{e_j} e_l, e_m) \\ = &0 \quad (\nabla_{e_i} e_j = 0 \text{ for } 1 \leq i, j \leq n \text{ and } R \text{ is a tensor. })\end{aligned}$$

Since the coderivative of R is still a tensor, then by the linearity of R, we have

$$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0$$

Problem 2

Recall the corollary in the lecture. It says sectional curvature $K(\sigma) \equiv c$ for all $\sigma \in T_p M$ if and only if $R(X, Y, Z, W) = c\left(\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle\right)$. So we have

$$R_p(X, Y, Z, W) = f(p)\left(\langle X, Z\rangle \langle Y, W\rangle - \langle Y, Z\rangle \langle X, W\rangle\right)$$

Again, we can work at normal coordinate. Let $e_1, \cdots, e_n$ to be the normal coordinate at p. Use the properties $\frac{\partial}{\partial x^i} \langle e_j, e_k\rangle = 0$ at p for any $1 \leq i, j, k \leq n$, we have

$$
\begin{aligned}
(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) &= \frac{\partial}{\partial x^i}\left(f(p)\left(\langle e_j, e_l\rangle \langle e_k, e_m\rangle - \langle e_j, e_m\rangle \langle e_k, e_l\rangle\right)\right) \\
&= \frac{\partial f}{\partial x^i}(p)\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)
\end{aligned}
$$

Now since $n \geq 3$, for any i, we can choose j, k such that i, j, k are all different with each other. Choose $l = j, m = k$ and use second Bianchi Identity, we have

$$
0 = \frac{\partial f}{\partial x^i}(1-0) + \frac{\partial f}{\partial x^j}(0-0) + \frac{\partial f}{\partial x^k}(0-0) = \frac{\partial f}{\partial x^i}
$$

So $\nabla f = 0$ at p. Since p is arbitrary, we know f is a constant function.

Problem 3

(a) Let $(x_1, \cdots, x_n)$ to be a local coordinate with $e_i = \frac{\partial}{\partial x_i}$. We write $R(e_i, e_j, e_k, e_l) = R_{ijkl}$ and $\mathrm{Ric}(e_i, e_j) = \mathrm{Ric}_{ij}$ for short. So our condition says $\mathrm{Ric}_{ij} = \lambda g_{ij}$. The second Bianchi identity can be written as

$$
\nabla_{e_i} R_{jklm} + \nabla_{e_j} R_{kilm} + \nabla_{e_k} R_{ijlm} = 0
$$

We multiply g^{jl}, g^{km} to the both side of the above identity and take sum over j, l, k, m, and using coderivative of metric is 0, we have

$$
\begin{aligned}
0 &= g^{jl} g^{km} \nabla_{e_i} R_{jklm} + g^{jl} g^{km} \nabla_{e_j} R_{kilm} + g^{jl} g^{km} \nabla_{e_k} R_{ijlm} \\
&= \nabla_{e_i}\left(g^{jl} g^{km} R_{jklm}\right) + \nabla_{e_j}\left(g^{jl} g^{km} R_{kilm}\right) + \nabla_{e_k}\left(g^{jl} g^{km} R_{ijlm}\right) \\
&= \nabla_{e_i}\left(g^{jl}\mathrm{Ric}_{jl}\right) + \nabla_{e_j}\left(-g^{jl}\mathrm{Ric}_{il}\right) + \nabla_{e_k}\left(-g^{km}\mathrm{Ric}_{im}\right) \quad \text{(Definition of Ric)} \\
&= \nabla_{e_i}\left(g^{jl} g_{jl}\lambda\right) - \nabla_{e_j}\left(g^{jl} g_{il}\lambda\right) - \nabla_{e_k}\left(g^{km} g_{im}\lambda\right) \quad (\mathrm{Ric}_{ij} = \lambda g_{ij}) \\
&= \nabla_{e_i}(n\lambda) - \nabla_{e_j}(\delta_i^j \lambda) - \nabla_{e_k}(\delta_i^k \lambda) = (n-2)\frac{\partial}{\partial x_i}\lambda
\end{aligned}
$$

where we've used Einstein summation convention.

Hence $\nabla \lambda \equiv 0$ on M. So λ is a constant function since M is connected.

(b) Let e_1, e_2 be any orthogonal vectors at p. So the section curvature at the plane spanned by e_1, e_2 is R_{1212}. Let's choose e_3 to form a orthonormal basis of T_pM with e_1, e_2 and extend them to a local frame. Note that

$$
\begin{aligned}
&\mathrm{Ric}_{11} + \mathrm{Ric}_{22} - \mathrm{Ric}_{33} \\
&= R_{1212} + R_{1313} + R_{2121} + R_{2323} - R_{3131} - R_{3232} \\
&= 2R_{1212} = K(e_1, e_2)
\end{aligned}
$$

Note that $\mathrm{Ric}_{ii} = \lambda \langle e_i, e_i\rangle = \lambda$, we have $K(e_1, e_2) = \lambda$ for any point p and any $e_1, e_2 \in T_pM$, with e_1, e_2 the normal orthogonal vectors at p.

So M has constant sectional curvature.

Problem 4

Given $p \in \Sigma$, choose a orthonormal basis $\{e_1, \cdots, e_{n-1}\}$ of $T_p\Sigma$ at p. So the vectors $\{e_1, \cdots, e_{n-1}, N\}$ will form a orthonormal basis of T_pM. The mean curvature H of Σ with respect to N is defined as

$$H = \sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, N \rangle$$

where we've extend $\{e_i\}$ to any local frame of Σ and $\nabla_X Y$ denote the coderivative on M. Since $\langle e_i, N \rangle \equiv 0$ on Σ, we have

$$H = \sum_{i=1}^{n-1} e_i \langle e_i, N \rangle - \sum_{i=1}^{n-1} \langle e_i, \nabla_{e_i} N \rangle = - \sum_{i=1}^{n-1} \langle e_i, \nabla_{e_i} N \rangle$$

Note that $\langle N, N \rangle = 1$ all the time. So

$$0 = N \langle N, N \rangle = 2 \langle N, \nabla_N N \rangle$$

Hence

$$H = - \sum_{i=1}^{n-1} \langle \nabla_{e_i} e_i, N \rangle - \langle \nabla_N N, N \rangle = -\mathrm{div} N = \mathrm{div} \frac{\nabla f}{|\nabla f|}.$$

(There might be a sign difference based on how to define the mean curvature and how to choose the normal.)

Problem 5

(a) We note the following identity

$$F_*(\frac{\partial}{\partial u}) = \frac{\partial F}{\partial u} = (-\sin u, \cos u, 0, 0),$$
$$F_*(\frac{\partial}{\partial v}) = \frac{\partial F}{\partial v} = (0, 0, -\sin v, \cos v)$$

So

$$\left\langle F_*(\frac{\partial}{\partial u}), F_*(\frac{\partial}{\partial u}) \right\rangle_{\mathbb{R}^4} = 1 = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle_{\mathbb{R}^2}$$
$$\left\langle F_*(\frac{\partial}{\partial u}), F_*(\frac{\partial}{\partial v}) \right\rangle_{\mathbb{R}^4} = 0 = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle_{\mathbb{R}^2}$$
$$\left\langle F_*(\frac{\partial}{\partial v}), F_*(\frac{\partial}{\partial v}) \right\rangle_{\mathbb{R}^4} = 1 = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle_{\mathbb{R}^2}$$

Hence F is an isometric immersion.

(b) Note that $|F|^2 = \cos^2 u + \sin^2 u + \cos^2 v + \sin^2 v = 2$. So image of F lies inside $\mathbb{S}^3$.

Let N be a unit normal vector field along $F(\mathbb{R}^2)$ in $\mathbb{S}^3$. We use $\overline{\nabla}_X Y$ to denote the coderivative on $\mathbb{R}^4$ and $\nabla_X Y$ denote the coderivative on $\mathbb{S}^3$.

So the mean curvature can be calculate by following

$$\begin{aligned}
H &= \left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, N \right\rangle + \left\langle \nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v}, N \right\rangle \\
&= \left\langle \overline{\nabla}_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, N \right\rangle + \left\langle \overline{\nabla}_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v}, N \right\rangle \quad (\text{ since } N \in T\mathbb{S}^3) \\
&= \left\langle \frac{\partial^2 F}{\partial u^2}, N \right\rangle + \left\langle \frac{\partial^2 F}{\partial v^2}, N \right\rangle \\
&= \langle (-\cos u, -\sin u, 0, 0) + (0, 0, -\cos v, -\sin v), N \rangle = -\langle F(u,v), N \rangle \\
&= 0 \quad (\text{ since } N \perp F \in \mathbb{R}^4)
\end{aligned}$$

So Σ is a minimal immersion into $\mathbb{S}^3$.