MATH 5061 Riemannian Geometry

Solution to Problem Set 3

Problem 1

Firstly, note that antipodal map $A(p) = -p$ will give an isometry on $\mathbb{R}^{n+1}$. That is, let g be the metric on $\mathbb{R}^{n+1}$, then

$$\begin{aligned}(A^*g)_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= g_{-p}(dA_p(\frac{\partial}{\partial x_i}), dA_p(\frac{\partial}{\partial x_j})) \\ &= g_{-p}(-\frac{\partial}{\partial x_i}, -\frac{\partial}{\partial x_j}) = \delta_{ij} \\ &= g_{-p}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\end{aligned}$$

So $A^*g = g$. Hence $A^*(g|_{\mathbb{S}^n}) = g|_{\mathbb{S}^n}$, A will induce an isometry on $\mathbb{S}^n$.

Now we have the nature definition of metric $\tilde{g}$ on $\mathbb{RP}^n$ defined by

$$\tilde{g}_q(v, w) = g_p|_{\mathbb{S}^n}(v_0, w_0)$$

where $q \in \mathbb{RP}^n, p \in \pi^{-1}(q)$, $v_0 \in d\pi_p^{-1}(v), w_0 \in d\pi_p^{-1}(w)$. Note that v_0, w_0 is uniquely determined by v, w since $d\pi_p$ is a isomorphism. We need to verity $\tilde{g}$ is well-defined.

If p' is another p such that $\pi(p') = q$, then $p' = -p = A(p)$. Hence $g_p|_{\mathbb{S}^n}(v_0, w_0) = g_{A(p)}|_{\mathbb{S}^n}(dA_p(v_0), dA_p(w_0))$. Note that $d\pi_{A(p)} \circ dA_p = \pi_p$ by $\pi \circ A = \pi$, so p' will give the same definition with p.

By the construction above, we can find π is indeed a local isometry since locally they are diffeomorphism and their metric is related by π.

Problem 2

Let $\mathcal{F} := \{F : \mathbb{S}^n \to \mathbb{S}^n | F \text{ is an isometry }\}$. Then we know $O(n+1) \subset \mathcal{F}$ since the orthogonal transformation will keep the metric of $\mathbb{R}^{n+1}$ and hence keep the metric on $\mathbb{S}^n$.

We will show that $O(n+1) = \mathcal{F}$.

Let $\varphi \in \mathcal{F}$ be an isometry of $\mathbb{S}^n$. Then we construct a new map $\psi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{n+1}\backslash\{0\}$ in the following ways

$$\psi(x) = |x|\, \varphi(\frac{x}{|x|}), x \in \mathbb{R}^{n+1}\backslash\{0\}.$$

One can verify this is a diffeomorphism. Moreover, we can calculate the differential map at x with direction v as following, (e.g. calculating $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\psi(c(t))$

with $c(0) = x, c'(0) = v$)

$$
\begin{aligned}
d\psi_x(v) &= \varphi\left(\frac{x}{|x|}\right)\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\,|x+tv| + |x|\,\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi\left(\frac{x+tv}{|x+tv|}\right) \\
&= \frac{\langle x, v\rangle}{|x|^2}\varphi\left(\frac{x}{|x|}\right) + |x|\,d\varphi_{\frac{x}{|x|}}\left(\frac{v}{|x|} - \frac{\langle x, v\rangle\, x}{|x|^3}\right)
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^{n+1}$ (Or the standard metric on Euclidean space)

Use the fact φ is an isometry, i.e. $\langle d\phi_p(v), d\phi_p(w)\rangle = \langle v, w\rangle$, and then fact $\varphi(\frac{x}{|x|})\perp \operatorname{Im}\left(d\phi_{\frac{x}{|x|}}\right), \left|\varphi(\frac{x}{|x|})\right| = 1$, we find

$$
\begin{aligned}
\langle d\psi_x(v), d\psi_x(w)\rangle &= \frac{\langle x, v\rangle\langle x, w\rangle}{|x|^4} + |x|^2\left\langle \frac{v}{|x|} - \frac{\langle x, v\rangle\, x}{|x|^3}, \frac{w}{|x|} - \frac{\langle x, w\rangle\, x}{|x|^3}\right\rangle \\
&= \langle v, w\rangle\,.
\end{aligned}
$$

So we get $\psi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{n+1}\backslash\{0\}$ is an isometry. Now we can use the properties of Euclidean space to show ψ is indeed a linear map.

Since ψ is an isometry, it keeps the distance of different points. That is, if $p, q \in \mathbb{R}^{n+1}\backslash\{0\}$, such that the line segment pq doesn't contain 0, then $|\psi(p) - \psi(q)| = |p - q|$. If the line segment pq contains 0, since ψ is continuous, we still have the same result since we can choose $q_i \to q$ such that pq_i does not contain 0 and take limit in $|\psi(p) - \psi(q_i)| = |p - q_i|$.

Again, by the definition of ψ, we know ψ keeps the length of points. That is

$$
|\psi(p)| = |p|\left|\varphi(\frac{p}{|p|})\right| = |p|\,.
$$

Hence ψ keeps the inner product by the following

$$
\begin{aligned}
\langle\psi(p), \psi(q)\rangle = \frac{1}{2}\left(|\psi(p)|^2 + |\psi(q)|^2 - |\psi(p) - \psi(q)|^2\right) &= \frac{1}{2}\left(|p|^2 + |q|^2 - |p - q|^2\right) \\
&= \langle p, q\rangle
\end{aligned}
$$

for any $p, q \in \mathbb{R}^{n+1}\backslash\{0\}$.

So for any $a, b \in \mathbb{R}, p, q, r \in \mathbb{R}^{n+1}\backslash\{0\}$, we have

$$
\langle\psi(ap + bq) - a\psi(p) - b\psi(q), \psi(r)\rangle = \langle ap + bq, r\rangle - a\langle p, r\rangle - b\langle q, r\rangle = 0
$$

Note that $\psi(r)$ can take any vectors in $\mathbb{S}^n$, by choose $\psi(r) = e_1, \cdots, e_{n+1}$ to be the basis of $\mathbb{R}^{n+1}$, we actually know

$$
\psi(ap + bq) = a\psi(p) + b\psi(q).
$$

Hence if we define $\psi(0) = 0$, we actually get $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear map. It is an orthogonal map since ψ also keeps the length of any line segments of $\mathbb{R}^{n+1}$.

So as a restriction of ψ, the map φ is an orthogonal transformation on $\mathbb{S}^n$.

Problem 3

(a). Let write a new curve $\tilde{c}(s) = c(t + t_0 - s)$ from $c(t)$ to $c(t_0)$ for $s \in [t_0, t]$. So we can define the new map $\tilde{P} = P_{\tilde{c},t,t_0} : T_{c(t)}M \to T_{c(t_0)}M$.

Note $P, \tilde{P}$ are all homomorphism since for constant a, b, we always have $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$.

Let's show $\tilde{P} \circ P = \mathrm{Id}_{T_{c(t_0)}M}$. This is because, for any $V(c(t))$, the parallel transportation of $V \in T_{c(t_0)}M$ along c, we consider the vector fields $V(c(s)) = V(\tilde{c}(t + t_0 - s))$, we have

$$\nabla_{\tilde{c}'(s)} V = \nabla_{-c'(s)} V = 0$$

Hence $V(\tilde{c}(t + t_0 - s))$ is a parallel transport from $V(c(t))$ along $\tilde{c}$. Hence $\tilde{P}(V(c(t))) = V(c(t_0))$. That's $\tilde{P} \circ P(V(c(t_0))) = V(c(t_0))$.

Similarly, we know $P \circ \tilde{P} = \mathrm{Id}_{T_{c(t)}}M$. Hence P is an isomorphism.

For the linear isometry, Let V, W be two vectors fields that are all paralleled along c. Since the metric is compatible with connection, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} g(V(s), W(s)) &= g(\nabla_{c'(s)} V(s), W(s)) + g(V(s), \nabla_{c'(s)} W(0)) \\ &= g(0, W(s)) + g(V(s), 0)) = 0\end{aligned}$$

Integrate s from t_0 to t, we have $g(V(t), W(t)) = g(V(t_0), W(t_0))$.

If M is orientable, we consider the $P_s = P_{c,t_0,s}$ for any $s \in [t_0, t]$. Let's choose an orientable basis $e_1, \cdots e_n \in T_{c(t_0)}M$ and let $e_i(s) = P_s(e_i)$, the parallel transport of e_i along c.

Let's consider the function $f(s) : [t_0, t] \to \{-1, 1\}$ where $f(s) = 1$ if and only if P_s is orientation-preserving.

Clearly $f(s)$ is continuous since in any oriented local coordinate chart $x_1, \cdots, x_n$, we write $e_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j}$, then orientation of $e_i(s)$ is determined by the sign of $\det(a_{ij}(s))$, which is continuous with respect to s.

Since $f(t_0) = 1$, we get $f(s) = 1$ for all $s \in [t_0, t]$. So P is orientation preserving.

(b).

As before, we choose $e_1, \cdots, e_n$ as the basis of $T_{c(t_0)}M$, and let $e_i(c(t))$ be the parallel transformation along $c(t)$ from the vectors e_i. Since $e_i(c(t))$ is the basis of $T_{c(t)}M$ by the isomorphism of P, we can write $Y(c(t)) = a_i(t)e_i(c(t))$. Hence

$$\begin{aligned}\nabla_X Y(p) &= \sum_{i=1}^{n} \nabla_{c'(0)}(a_i(t)e_i(c(t)))|_{t=t_0} = \sum_{i=1}^{n} c'(0)(a_i(t))e_i(p) + a_i(0)\nabla_{c'(0)} e_i(p) \\ &= \sum_{i=0}^{n} a_i'(0)e_i(p)\end{aligned}$$

Here $c'(0)(a_i(t))$ means the vector $c'(0)$ acting on the function $a_i(t)$.

On the other hand, use the fact that $P^{-1}_{c,t_0,t}$ is a linear map, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t))) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \sum_{i=1}^{n} a_i(t) P^{-1}_{c,t_0,t}(e_i(c(t))) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \sum_{i=1}^{n} a_i(t) e_i(c(t_0)) \\ &= \sum_{i=0}^{n} a_i'(0) e_i(p)\end{aligned}$$

Hence $(\nabla_X Y)(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t)))$.

Problem 4

(a). Let work at the local coordinate $(x_1, \cdots, x_n)$ near p. Then TM has the local coordinate $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ near (p, v) defined by

$$(p, v) = ((p_1, \cdots, p_n), (v_1 \frac{\partial}{\partial x_1}, \cdots, v_n \frac{\partial}{\partial x_n})) \to (p_1, \cdots, p_n, v_1, \cdots, v_n)$$

So for $\alpha(t)$, if $v(t) = v_1(t)\frac{\partial}{\partial x_1} + \cdots + v_n(t)\frac{\partial}{\partial x_n}$, $p(t) = (p_1(t), \cdots p_n(t))$, then $\alpha(t)$ can be represented by $(p_1, \cdots, p_n, v_1, \cdots, v_n)$. Hence $\alpha'(t) = p_1'(t)\frac{\partial}{\partial x_1} + \cdots + p_n'(t)\frac{\partial}{\partial x_n} + v_1'(t)\frac{\partial}{\partial y_1} + \cdots + v_n'(t)\frac{\partial}{\partial y_n}$. So we know that $p_i'(0)$ and $v_i'(0)$ are uniquely determined by V.

Note that π has the form $(x_1, \cdots, x_n, y_1, \cdots, y_n) \to (x_1, \cdots, x_n)$ under our local coordinates, so $d\pi(\alpha'(0)) = p_1'(0)\frac{\partial}{\partial x_1} + \cdots p_n'(0)\frac{\partial}{\partial x_n}$. Hence $d\pi(V) = d\pi(\alpha'(0))$ will be determined by V, which does not rely on the choice of curve $(p(t), v(t))$.

For the second part, we have

$$\frac{Dv}{dt}(0) = \nabla_{p'(0)} v(t) = \sum_{i=1}^{n} v_i'(0)\frac{\partial}{\partial x_i} + v_i(0)\nabla_{p'(0)}\frac{\partial}{\partial x_i}$$

Since $v_i'(0)$ is uniquely determined by V, $v_i(0)$ is uniquely determined by v, $p'(0) = \pi(V)$ is uniquely determined by V, we know $\frac{Dv}{dt}(0)$ does not rely on the choice of curves.

Hence all of the terms in the definition of $\langle V, W\rangle_{(p,v)}$ doesn't rely on the choice of curves and hence it indeed give us a Riemannian metric on TM.

Moreover, we have the description of inner product on TM as following. If $V = (\tilde{p}, \tilde{v}) = (\tilde{p}_1, \cdots, \tilde{p}_n, \tilde{v}_1, \cdots, \tilde{v}_n), W = (\tilde{q}, \tilde{w}) = (\tilde{q}_1, \cdots, \tilde{q}_n, \tilde{w}_1, \cdots, \tilde{w}_n) \in T_{(p,v)}TM$, we have

$$\langle V, W\rangle_{p,v} = \langle \tilde{p}, \tilde{q}\rangle_p + \left\langle \tilde{v} + \sum_{i,j,k=1}^{n} v_i \tilde{p}_j \Gamma_{ij}^k \frac{\partial}{\partial x_k}, \tilde{w} + \sum_{i,j,k=1}^{n} w_i \tilde{q}_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} \right\rangle_p$$

where $\tilde{v}$ means the canonical projection when viewed it as a vector $\sum_{i=1}^{n} \tilde{v}_i \frac{\partial}{\partial x_i}$ in T_M.

(b). A vector $(p, v) = (p_1, \cdots, p_n, v_1, \cdots, v_n)$ is in the fiber $\pi^{-1}(p)$ if the projection $d\pi(p, v) = 0$. This means the vectors in the fiber $\pi^{-1}(p)$ is spanned by all the vector having form $(0, v)$ for $v \in T_pM$.

Hence $c(t)$ is horizontal $\iff \langle (p(t), v(t)), (0, w(s)) \rangle_{(p(t_0), v(t_0))}$ for each $w(s) \in T_{p(t_0)}M$, a curve of vectors in $T_{p(t_0)}M$ and each $t_0 \iff \left\langle \frac{Dv}{dt}, \frac{Dw}{ds} \right\rangle = 0$ at t_0. Note that for any $w_0 \in T_{p(t_0)}M$, we can choose some special curve $w(s)$ such that $\frac{Dw}{ds} = w_0$. Hence the above equivalent to the fact that $\frac{Dv}{dt}(t_0) = 0$ for any t_0, which equivalent to the fact $v(t)$ is a parallel vector field along $p(t)$.
(c). Locally, the geodesic fields at $(p_1, \cdots, p_n, v_1, \cdots, v_n)$ is defined by

$$V = (\tilde{p}, \tilde{v}) := \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} + \sum_{k=1}^{n} \sum_{i,j=1}^{n} -\Gamma_{ij}^{k} v_i v_j \frac{\partial}{\partial y_k}$$

In the local description in **(a)**, for any $W = (0, \tilde{w}) \in T_{(p,v)}TM$, we have

$$\begin{aligned} \langle V, W \rangle_{(p,v)} &= \left\langle \sum_{k=1}^{n} \left(\tilde{v}_k + \sum_{i,j=1}^{n} v_i \tilde{p}_j \Gamma_{ij}^{k} \right) \frac{\partial}{\partial x_k}, \tilde{w} \right\rangle \\ &= \left\langle \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} -\Gamma_{ij}^{k} v_i v_j + \sum_{i,j=1}^{n} v_i v_j \Gamma_{ij}^{k} \right), \tilde{w} \right\rangle = 0 \end{aligned}$$

Hence V is a horizontal vector field.
(d). Let $c(t) = (p(t), v(t))$ be any curves in TM. So we know $c(t)$ is the trajectories of the geodesic field if and only if $p'(t) = v(t)$ and $\frac{Dv}{dt} = 0$, this will equivalent to the fact $p(t)$ is a geodesic in M.

So we only need to show any geodesic $p(t) : I \to M$ of M can be lifted to TM such that $c(t) := (p(t), p'(t))$ is a geodesic on TM. Let's suppose $I = [0, T]$ for convenience.

Let $\tilde{c}(t), t \in [0, \varepsilon']$ be a shortest geodesic joining $c(0), c(\varepsilon)$ in TM for some small ε. Let $\tilde{c}(t) = (\tilde{p}(t), \tilde{v}(t))$, then $\tilde{p}$ is a curve from $p(0)$ to $p(\varepsilon)$.

Let L be the length of $p(t), t \in [0, \varepsilon]$, $\tilde{L}$, the length of $\tilde{p}(t), t \in [0, \varepsilon']$. So for ε small enough, we know $\tilde{L} \geq L$, since geodesics are locally minimizing the length. Now let's calculate the length of $c, \tilde{c}$.

Note that

$$\begin{aligned} \int_0^{\varepsilon} |c'(t)|\, dt &= \int_0^{\varepsilon} \sqrt{|p'|^2 + \left| \frac{Dp'}{dt} \right|^2}\, dt = \int_0^{\varepsilon} |p'|^2\, dt = L \\ \int_0^{\varepsilon'} |\tilde{c}'(t)|\, dt &= \int_0^{\varepsilon'} \sqrt{|\tilde{p}'|^2 + \left| \frac{Dv}{dt} \right|^2}\, dt \geq \int_0^{\varepsilon'} |\tilde{p}'|\, dt = \tilde{L} \geq L \end{aligned}$$

But since we've assumed $\tilde{c}$ are the shortest geodesic joining two points, we have $\int_0^{\varepsilon'} |\tilde{c}'|\, dt \leq \int_0^{\varepsilon} |c'|\, dt$. Hence all of inequalities above are indeed equalities. So $c, \tilde{c}$ are indeed the same curves. This shows c is also the geodesic in TM. Hence, we finished our proof.