MATH 5061 Riemannian Geometry

Solution to Problem Set 2

Problem 1

When we restrict f on 2-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, we will get a map $G(x, y, z) = F|_{\mathbb{S}^2}(x, y, z) = (x^2 - y^2, xy, xz, yz)$.

Note that G takes the same values on antipodal points. That is, $G(-x, -y, -z) = ((-x)^2 - (-y)^2, (-x)(-y), (-x)(-z), (-y)(-z)) = G(x, y, z)$. So G will induce a map $\tilde{G} : \mathbb{S}^2/\sim = \mathbb{RP}^2 \to \mathbb{R}^4$ where $p \sim q$ if and only if $p = -q$ for $p, q \in \mathbb{S}^2$.

Now let's verity $\tilde{G}$ is injective. Suppose $\tilde{G}([x_1, y_1, z_1]) = \tilde{G}([x_2, y_2, z_2])$. Since $x_1^2 + y_1^2 + z_1^2 = 1$, we can consider three cases $x_1 \neq 0, y_1 \neq 0, z_1 \neq 0$.

For the first case $x_1 \neq 0$, let $t = \frac{x_2}{x_1}$. From $x_1 y_1 = x_2 y_2$ we have $y_1 = t y_2$. Put them into $x_1^2 - y_1^2 = x_2^2 - y_2^2$, we have $(1 - t^2)(x_1^2 + y_2^2) = 0 \implies t = \pm 1$. If $t = 1$,, then $x_1 = x_2, y_1 = y_2 \implies z_1 = z_2$. For $t = -1$, we will have $x_1 = -x_2, y_1 = -y_2, z_1 = -z_2$. No matter what, we have $[(x_1, y_1, z_1)] = [(x_2, y_2, z_2)]$.

For the other cases $y_1 \neq 0, y_2 \neq 0$, we have the similar argument to show $[(x_1, y_1, z_1)] = [(x_2, y_2, z_2)]$.

So we get $\tilde{G}$ is indeed injective. This shows $\tilde{G}$ is homeomorphic to its image by the result in Topology.

Now, we need to check $\tilde{G}$ is immersion. Note that the quotient map $\pi : \mathbb{S}^2 \to \mathbb{RP}^2$ is locally diffeomorphism. So we only need to verify $G : \mathbb{S}^2 \to \mathbb{R}^4$ is an immersion.

Fix any $p_0 = (x_0, y_0, z_0) \in \mathbb{S}^2$. Since $\mathbb{S}^2$ is a submanifold of $\mathbb{R}^3$, we can identify the $T_{p_0}(\mathbb{S}^2)$ with the subspace of $\mathbb{R}^3$. Note that $T_{p_0}(\mathbb{S}^2)$ should perpendicular to p_0, and hence $T_{p_0}(\mathbb{S}^2)$ can be spanned any two non-zero vectors of the vectors $X_1 = (-y_0, x_0, 0), X_2 = (0, -z_0, y_0), X_3 = (-z_0, 0, x_0)$. In order to proof G is an immersion, we need to show $\{dG(X_1), dG(X_2), dG(X_3)\}$ spans a space at least dimension 2 in $T_{G(p_0)}\mathbb{R}^4 \simeq \mathbb{R}^4$.

Let $\iota : \mathbb{S}^2 \to \mathbb{R}^3$ be the immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$. So $G = F \circ \iota$, $dG = dF \circ d\iota$. Note that we already know $\mathrm{Im}(d\iota)$ spanned by X_1, X_2, X_3 by above tangent space identification.

Direct calculation of dF at (x, y, z) gives us

$$dF = \begin{bmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}$$

Hence

$$\begin{aligned} dF_{p_0}(X_1) &= (-4x_0y_0, x_0^2 - y_0^2, -y_0z_0, x_0z_0) \\ dF_{p_0}(X_2) &= (2y_0z_0, -x_0z_0, x_0y_0, y_0^2 - z_0^2) \\ dF_{p_0}(X_3) &= (-x_0z_0, -y_0z_0, x_0^2 - z_0^2, x_0y_0) \end{aligned}$$

It's easy to verify at least two of above vectors are linearly independent provided $x_0^2 + y_0^2 + z_0^2 = 1$.

Combining that $\tilde{G}$ is a homeomorphism into its image and an immersion, we know $\tilde{G} : \mathbb{RP}^2 \to \mathbb{R}^4$ is indeed an embedding.

Problem 2

For any $f \in C^\infty(M)$, we directly compute,

$$\begin{aligned} [X, [Y, Z]]f &= X([Y, Z]f) - [Y, Z](Xf) \\ &= X(YZf - ZYf) - YZXf + ZYXf \\ &= XYZf - YZXf + XZYf - ZYXf \end{aligned}$$

Similarly

$$\begin{aligned} [Y, [Z, X]]f &= YZXf - ZXYf + YXZf - XZYf \\ [Z, [X, Y]]f &= ZXYf - XYZf + ZYXf - YXZf \end{aligned}$$

Adding them up

$$\begin{aligned} &[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]f \\ = &(XYZ + YZX + ZXY)f - (YZX + ZXY + XYZ)f \\ &(XZY + YXZ + ZYX)f - (ZYX + XZY + YXZ)f \\ = &0 \end{aligned}$$

Hence

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

Problem 3

Let $f \in C^\infty(N)$ be any smooth function on N. For any $X \in \Gamma(TM)$, we will have

$$(\phi_* X)(f) \circ \phi = X(f \circ \phi)$$

This is because for any $p \in M$, $X_p(f \circ \phi) = \phi_*(X_p)(f)$, and note $\phi_*(X_p)$ is a vector at $\phi(p)$, we have $\phi_*(X_p)(f) = \phi_* X(f)(\phi(p))$, which is the what we want.

Hence,

$$(\phi_* X)(\phi_* Y)(f) \circ \phi = X((\phi_* Y)(f) \circ \phi) = X(Y(f \circ \phi)) = XY(f \circ \phi)$$

So

$$([\phi_* X, \phi_* Y]f) \circ \phi = XY(f \circ \phi) - YX(f \circ \phi) = [X, Y](f \circ \phi) = \phi_*[X, Y](f) \circ \phi$$

Note that ϕ is a diffeomorphism, so we have

$$[\phi_* X, \phi_* Y]f = \phi_*[X,Y](f)$$

as a function on N. Hence $[\phi_* X, \phi_* Y] = \phi_*[X,Y]$.

Now let $\{\phi_t\}_{t\in\mathbb{R}}$ be the flow generated by Y. If Y is not compactly supported, we will only require $\{\phi_t\}_{t\in(-\varepsilon,\varepsilon)}$ defined near a fixed point p. Since ϕ_t is a local diffeomorphism near p, we have

$$(\phi_t)_*[Z,X] = [(\phi_t)_* Z, (\phi_t)_* X]$$

Take derivative with respect to t at $t=0$, we will have

$$[[Z,X],Y] = [[Z,Y],X] + [Z,[X,Y]]$$

at p where we've used the definition of derivative and right hand side comes from by inserting a middle term in the limit.

Using $[X,Y]=[Y,X]$, we will have

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Problem 4

By definition of pull-back, we have

$$(\phi_t^*\alpha)(Y_1,\cdots,Y_q)(x) = \alpha_{\phi_t(x)}(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q)$$

with $x\in M$ where ϕ_t is the flow generated by X.

So

$$\begin{aligned}
&(\mathcal{L}_X\alpha)(Y_1,\cdots,Y_q)(x)\\
=&\lim_{t\to 0}\frac{1}{t}\left((\phi_t^*\alpha)(Y_1,\cdots,Y_q)(x) - \alpha_x(Y_1,\cdots,Y_q)\right)\\
=&\lim_{t\to 0}\frac{1}{t}\left(\alpha_{\phi_t(x)}(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q) - \alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_q)\right)\\
&+\sum_{i=1}^{q}\lim_{t\to 0}\frac{1}{t}\left[\alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_{i-1},\phi_{t*}Y_i,Y_{i+1},\cdots,Y_q)\right.\\
&-\left.\alpha_x(\phi_{t*}Y_1,\cdots,\phi_{t*}Y_{i-1},Y_i,Y_{i+1},\cdots,Y_q)\right]\\
=&X(\alpha(Y_1,\cdots,Y_q))(x) + \sum_{i=1}^{q}\alpha_x(Y_1,\cdots,Y_{i-1},\mathcal{L}_X Y_i,Y_{i+1},\cdots,Y_q)\\
=&X(\alpha(Y_1,\cdots,Y_q))(x) - \sum_{i=1}^{q}\alpha_x(Y_1,\cdots,Y_{i-1},[X,Y_i],Y_{i+1},\cdots,Y_q)
\end{aligned}$$

Since the above identity holds for all $x\in M$, we have

$$(\mathcal{L}_X\alpha)(Y_1,\cdots,Y_q) = X(\alpha(Y_1,\cdots,Y_q)) - \sum_{i=1}^{q}\alpha(Y_1,\cdots,Y_{i-1},[X,Y_i],Y_{i+1},\cdots,Y_q)$$

Problem 5

(a). We write $\alpha = \iota^*\left(\sum_{k=0}^n dz^k \wedge d\overline{z}^k\right)$, the 2-form on $\mathbb{S}^{2n+1}$.

For any $y \in \mathbb{CP}^n$, we can find $x \in \mathbb{S}^{2n+1}$ such that $p(x) = y$. Since p is a projection and $dp : T_x\mathbb{S}^{2n+1} \to T_y\mathbb{CP}^n$ is surjective, we know if such ω exist, then it is completely determined by α. Namely, for any $Y_1, Y_2 \in T_y\mathbb{CP}^n$, we can find $X_1, X_2 \in T_x\mathbb{S}^{2n+1}$ such that $dp(X_j) = Y_j, j = 1, 2$. Then we define $\omega_y(Y_1, Y_2) = \alpha_x(X_1, X_2)$.

But before that, we need to verify the above definition doesn't depend on the choice of X_j and the point x.

First, we claim the following statement. If $X \in T_x\mathbb{S}^{2n+1}$ with $dp(X) = 0$ in $T_y\mathbb{RP}^n$, then $\alpha_x(X, X_1) = 0$ for any $X_1 \in T_x\mathbb{S}^{2n+1}$.

As a corollary, this will imply the definition of ω_y does not rely on the choice of X_j.

To prove the above statement, let's identify the $T_x\mathbb{S}^{2n+1}$ with the subspace of $T_x\mathbb{C}^{n+1} \simeq \mathbb{C}^{n+1}$. $T_x\mathbb{C}^{n+1}$ has a canonical basis $\{\partial_{x_0}, \cdots, \partial_{x_n}, \partial_{y_1}, \cdots, \partial_{y_n}\}$ where we use $\partial_* := \frac{\partial}{\partial *}$. But for convenience, we use $\mathcal{A} = \{\partial_{z_j}, \partial_{\overline{z}_j}\}_{0\leq j\leq n}$, the dual frame of $\{dz_j, d\overline{z}_j\}_{0\leq i\leq n}$, as our basis for $T_x\mathbb{C}^{n+1} \simeq \mathbb{C}^{n+1}$. Note that the inner product under this basis is as following

$$\langle \partial_{z_j}, \partial_{\overline{z}_k}\rangle = \frac{1}{2}\delta_{jk}, \langle \partial_{z_j}, \partial_{z_k}\rangle = 0, \langle \partial_{\overline{z}_j}, \partial_{\overline{z}_k}\rangle = 0.$$

Suppose $x = (z_0, \cdots, z_n)$. With the basis $\mathcal{A}$, we can write $x = \sum_{i=0}^n z_j\partial_{z_j} + \overline{z}_j\partial_{\overline{z}_j}$.

By the geometric property of sphere, the tangent space $T_x\mathbb{S}^{2n+1}$ is just the subspace containing the vectors perpendicular to position vector x. That is

$$T_x\mathbb{S}^{2n+1} = \{X \in T_x\mathbb{C}^{n+1} : \langle X, x\rangle = 0\}.$$

Let's define a new vector V_x related to x by $V_x = i\sum_{j=0}^n z_j\partial_{z_j} - \overline{z}_j\partial_{\overline{z}_j}$. (Here by multiple a complex number i, we make V_x to be a real vector over $\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}$.) By a simple calculation, we'll find $\langle V_x, x\rangle = \sum_{j=0}^n -\frac{1}{2}z_j\overline{z}_j + \frac{1}{2}z_j\overline{z}_j = 0$. So $V_x \in T_x\mathbb{S}^{2n+1}$.

Now let's show that $dp(V_x) = 0$. Choose a curve $\gamma(t) = (\cos t)x + (\sin t)V_x$. If one write $\gamma(t)$ in the standard coordinate, one can get $\gamma(t) = (e^{it}z_0, \cdots, e^{it}z_n) \in \mathbb{S}^{2n+1}$. Clearly $p \circ \gamma(t) = [(z_0), \cdots, z_n]$, which does not rely on t. So $\frac{\mathrm{d}}{\mathrm{d}t}p \circ \gamma = 0$. Hence $dp(V_x) = dp(\gamma'(0)) = 0$.

Note that $dp : T_x\mathbb{S}^{2n+1} \to T_y\mathbb{CP}^n$ is surjective, so $\dim\ker(dp) = 1$. Hence we know $\ker(dp)$ is spanned by V_x and moreover, we get an isomorphism $\tilde{T}_x\mathbb{S}^{2n+1} \to T_y\mathbb{CP}^n$, where

$$\tilde{T}_x\mathbb{S}^{2n+1} = \{X \in T_x : \langle X, V_x\rangle = 0\}$$

the space containing vectors perpendicular to x and V_x.

Finally, we need to show $\alpha_x(V_x, X) = 0$ for any $X \in T_x\mathbb{S}^{2n+1}$. We decompose X as $X = X_1 + aV_x$, where $a = \langle X, V_x\rangle$ and $X_1 \in \tilde{T}_x\mathbb{S}^{2n+1}$. Note that $\alpha_x(V_x, V_x) = 0$, we only need to show $\alpha_x(V_x, V_1) = 0$.

Suppose $X_1 = \sum_{j=0}^n a_j\partial_{z_j} + b_j\partial_{\overline{z}_j}$ in the coordinate $\mathcal{A}$. Then we have the following results.

$$\langle X_1, x\rangle = 0 \implies \frac{1}{2}\left(\sum_{j=0}^{n} a_j\overline{z}_j + b_j z_j\right) = 0,$$
$$\langle X_1, V_x\rangle = 0 \implies \frac{1}{2}\left(\sum_{j=0}^{n} -a_j\overline{z}_j + b_j z_j\right) = 0,$$
$$\implies \sum_{j=0}^{n} a_j\overline{z}_j = \sum_{j=0}^{n} b_j z_j = 0.$$

So

$$\alpha_x(V_x, X_1) = \sum_{k=0}^{n} dz^k \wedge d\overline{z}^k(V_x, X_1)(\text{Push forward the vectors into } \mathbb{C}^{n+1})$$
$$= \sum_{k=0}^{n} -ia_k\overline{z}_k - ib_k z_k = 0 \quad (\text{By above identity}).$$

Thus, we finish the proof of our statement. So this shows the definition of $\omega_x(Y_1, Y_2) = \alpha_x(X_1, X_2)$ does not rely the choice of X_1, X_2.

Now we need to show the above definition doesn't rely on choice of x, too.

Suppose we have $\tilde{x} \in \mathbb{S}^{2n+1}$ with $p(\tilde{x}) = p(x) = y$. So we will have $\lambda x := (\lambda z_0, \cdots, \lambda z_n) = \tilde{x}$ for some $\lambda \in \mathbb{C}$. Since $x, \tilde{x} \in \mathbb{S}^{2n+1}$, we actually know $|\lambda| = 1$. Hence $\lambda = e^{it}$ for some $t \in \mathbb{R}$.

Let's define a diffeomorphism on $\mathbb{S}^{2n+1}$ by

$$\varphi_t(x) := e^{it}x = (e^{it}z_0, \cdots, e^{it}z_n)$$

It's not hard to verify that φ_t is indeed a diffeomorphism. (Moreover, φ_t is a flow generated by vector fields V_x.) We note that $p \circ \varphi_t = p$, so we have $dp_{\varphi_t(x)} \circ (d\phi_t)_x = dp_x$. So for any $X_1, X_2 \in T_x\mathbb{S}^{2n+1}$, noting $(d\phi_t)_x (X_i) = e^{it}X_i$, we have $dp_{\tilde{x}}(e^{it}X_j) = dp_x(X_j) = Y_j$. To prove the definition is independent of choice of x, we only need to show $\alpha_{\tilde{x}}(\lambda X_1, \lambda X_2) = \alpha_x(X_1, X_2)$ since those vectors have the same image under tangent map.

Let suppose $X_j = \sum_{k=0}^{n} a_k^{(j)}\partial_{z_k} + b_k^{(j)}\partial_{\overline{z}_k}$. Easy calculation shows $\lambda X_j = \sum_{k=0}^{n} \lambda a_k^{(j)}\partial_{z_k} + \overline{\lambda} b_k^{(j)}\partial_{\overline{z}_k}$. Hence,

$$\begin{aligned}
\alpha_{\tilde{x}}(\lambda X_1, \lambda X_2) &= \sum_{k=0}^{n} dz^k \wedge d\overline{z}^k(\lambda X_1, \lambda X_2)\\
&= \sum_{k=0}^{n} \lambda\overline{\lambda} a_k^{(1)} b_k^{(2)} - \overline{\lambda}\lambda b_k^{(1)} a_k^{(2)}\\
&= \sum_{k=0}^{n} a_k^{(1)} b_k^{(2)} - b_k^{(1)} a_k^{(2)}\\
&= \sum_{k=0}^{n} dz^k \wedge d\overline{z}^k(X_1, X_2)\\
&= \alpha_x(X_1, X_2)
\end{aligned}$$

Finally, we get the definition of ω does not rely on the choice of x.

This shows $\omega_y(Y_1, Y_2) = \alpha_x(X_1, X_2)$ gives us a well-defined form such that

$$p^*\omega = \iota^* \sum_{k=0}^{n} dz^k \wedge d\overline{z}^k$$

(b). Let $M = (m_{kl})_{0\leq k,l\leq n} \in U(n+1)$. So $\sum_{l=0}^{n} m_{kl}\overline{m}_{ls} = \delta_{ks}$.

The natural action φ_M related to M on $\mathbb{CP}^n$ defined as following

$$\varphi_M([z_0, \cdots, z_n]) = [(z_0, \cdots, z_n)M]$$

where $(z_0, \cdots, z_n)M$ should be understood as multiplication by matrix.

Let $\tilde{\varphi}_M(z_0, \cdots, z_n) = (z_0, \cdots, z_n)M$ be the natural action on $\mathbb{C}^{n+1}$, so φ_M is just induced by $\tilde{\varphi}_M$ using quotient map. Once again, one may need to verify φ_M is well-define. But that is not hart to see.

Let's show $\beta := \sum_{k=0}^{n} dz^k \wedge d\overline{z}^k$ is invariant under the action of $\tilde{\varphi}_M$. Let $(w_0, \cdots, w_n) = \tilde{\varphi}_M(x) = (z_0, \cdots, z_n)M$. So $\tilde{\varphi}_M^{-1}(w_0, \cdots, w_n) = (w_0, \cdots, w_n)M^{-1}$. Or we can write as $z_k = \sum_{l=0}^{n} w_l\overline{m}_{lk}$. So the pull-back form of β under $\tilde{\varphi}_M^{-1}$ is

$$\begin{aligned}
\sum_{k=0}^{n} dz^k \wedge d\overline{z}^k &= \sum_{k=0}^{n}\sum_{l,s=0}^{n} \overline{m}_{lk} dw^l \wedge (m_{sk} d\overline{w}^s) \\
&= \sum_{k,l,s=0}^{n} m_{sk}\overline{m}_{lk} dw^l \wedge d\overline{w}^s \\
&= \sum_{l,s=0}^{n} \delta_{sl} dw^l \wedge d\overline{w}^s \\
&= \sum_{l=0}^{n} dw^l \wedge d\overline{w}^l
\end{aligned}$$

Hence β is indeed invariant under the action of $\tilde{\varphi}_M$.

Note that by the construction of ω in (a), we know w is uniformly determined by β. So pass to the quotient map, we know ω should be invariant under the action of φ_M.

(c). Since ω is invariant under the natural action φ_M for $M \in U(n+1)$, we know that ω^k is also invariant under the action of φ_M since the pull-back of diffeomorphism keeps the wedge problem of differential form.

Now we show $\omega^k \in \Omega^{2k}\mathbb{CP}^n$ is non-zero. Actually we only need to show it is non-zero for $k = n$ since $\omega^{k_1} = 0 \implies \omega^{k_2} = 0$ for $k_2 \geq k_1$.

Since for any point $y \in \mathbb{CP}^n$, we can find a natural action φ_M such that $\varphi_M(y) = [(1, 0, \cdots, 0)]$ since $U(n+1)$ is transitive on sphere $\mathbb{S}^{2n+1}$. Using the invariant of ω under $U(n+1)$, we only need to show ω^n is non-zero at $[(1, 0, \cdots, 0)]$.

Let $x_0 = (1, 0, \cdots, 0)$. By the pull-back property, we have $p^*\omega^n = \alpha^n$. Note that when restricting $T_{x_0}\mathbb{S}^{2n+1} \simeq \{(y_0 i, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} : y_0 \in \mathbb{R}\}$, the term $dx^0 = 0$. So $\alpha_{x_0} = \sum_{k=1}^{n} z^k \wedge \overline{z}^k$. Hence

$$\alpha_{x_0}^n = n! z^1 \wedge \overline{z}^1 \wedge \cdots \wedge z^n \wedge \overline{z}^n$$

This is clearly non-zero. Hence $\omega^n_{p(x_0)}$ is non-zero.

This shows ω^k is non-zero for any $1 \le k \le n$.

Remark. In the view of Symplectic Geometry, the ω defined above is the canonical symplectic form on $\mathbb{CP}^n$. Since $\mathbb{CP}^n$ has a natural complex structure, together with this form and canonical metric, we can find $\mathbb{CP}^n$ is indeed a Kähler manifold. Although we didn't use the result coming from Symplectic Geometry explicitly, there are still many things borrowed from Symplectic Geometry.

Moreover, the ω is called **Fubini-Study Form**. Up to a constant, ω has form

$$\omega_{FS} = \left(\sum_{j=0}^{n} \frac{dz^j \wedge d\overline{z}^j}{\|z\|^2} - \sum_{j,k=0}^{n} \frac{\overline{z}_j z_k dz^j \wedge d\overline{z}^k}{\|z\|^4} \right)$$

where $\|z\|^2 = \sum_{k=0}^n |z_k|^2$ in homogeneous coordinates $[(z_0, \cdots, z_n)]$. In the coordinate chart $\{z_0 \neq 0\}$, $[(z_0, z_1, \cdots, z_n)] \to (w_1, \cdots, w_n) \in \mathbb{C}^n$, where $w_j = \frac{z_j}{z_0}$, ω_{FS} has the form

$$\omega_{FS} = \left(\sum_{k=1}^{n} \frac{dw^k \wedge d\overline{w}^k}{(1+|w|^2)} - \sum_{j,k=1}^{n} \frac{\overline{w}_j w_k dw^j \wedge d\overline{w}^k}{(1+|w|^2)^2} \right)$$

where $|w|^2 = \sum_{k=1}^n |w_k|^2$.

So if one can get the exact expression of ω in the local coordinate, there might be a direct proof for this problem. One only need to verify this definition does not rely the choice of coordinate so it indeed defines a global two form and calculating its pull-back to show it indeed agree with the relation in the problem.

One way to get the expression of ω_{FS} is to define the "inverse" map. Let $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{S}^{2n+1}, \pi(x) = \frac{x}{\|x\|}$ be the canonical projection of $\mathbb{S}^{2n+1}$(This is a vector bundle over $\mathbb{S}^{2n+1}$), the pull-back form of β will exact has expression listed above. Then on coordinate $\{z_0 \neq 0\}$, we just take $z_0 = 1$ to get the expression of ω in the local coordinate chart.

Remark. One can take $n = 1$ to get the understanding of above proof. For example, suppose $x = (a_0 + ib_0, a_1 + ib_1) = (a_0, b_0, a_1, b_1) \in \mathbb{S}^3$, then $V_x = (-b_0, a_0, -b_1, a_1)$. The space $\tilde{T}_x\mathbb{S}^3$ is spanned by $(-a_1, b_1, a_0, -b_0), (-b_1, -a_1, b_0, a_0)$. The dp_x is just like the projection from $T_x\mathbb{S}^3$ to $\tilde{T}_x\mathbb{S}^3$. The ω is just the volume form on $\mathbb{CP}^1 \simeq \mathbb{S}^2$ up to a constant.