THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050C Mathematical Analysis I

Tutorial 10 (March 27)

The following were discussed in the tutorial this week:

1 One-Sided Limits

Definition 1.1. Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$.

(i) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A : x > c\}$, then we say that $L \in \mathbb{R}$ is a **right-hand limit of f at** c and we write

$$\lim_{x \to c+} f = L \quad \text{or} \quad \lim_{x \to c+} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < x - c < \delta$, then $|f(x) - L| < \varepsilon$.

(ii) If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, c) = \{x \in A : x < c\}$, then we say that $L \in \mathbb{R}$ is a **left-hand limit of f at** c and we write

$$\lim_{x \to c-} f = L \quad \text{or} \quad \lim_{x \to c-} f(x) = L$$

if given any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for all $x \in A$ with $0 < c - x < \delta$, then $|f(x) - L| < \varepsilon$.

Theorem 1.1. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of $A \cap (c, \infty)$. Then the following statements are equivalent:*

(i) $\lim_{x \to c+} f = L$.

(ii) *For every sequence (x_n) that converges to c such that $x_n \in A$ and $x_n > c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Theorem 1.2. *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of both $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. Then $\lim_{x \to c} f = L$ if and only if $\lim_{x \to c+} f = L = \lim_{x \to c-} f$.*

Example 1.1. (a) $f(x) := \operatorname{sgn}(x)$

(b) $g(x) := e^{1/x}$ for $x \neq 0$.

(c) $h(x) := 1/(e^{1/x} + 1)$ for $x \neq 0$.

2 Coursework

Let $f(x) = \dfrac{e^{1/x}}{e^{2/x} + 1}$ for $x \neq 0$. Evaluate $\lim_{x \to 0+} f$ and $\lim_{x \to 0-} f$. Does $\lim_{x \to 0} f$ exist?