MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 9

4.2-4

Denote $f(x) := \cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$, $a_n = \frac{1}{2n\pi}, n \in \mathbb{N}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} f(x) = L$ exist, which implies that $L = \lim(f(a_n)) = \lim(f(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $f(a_n) = \cos 2n\pi = 1$ and $f(b_n) = \cos(2n+\frac{1}{2})\pi = 0$ for any $n \in \mathbb{N}$. Thus $\lim f(a_n) = 1$ while $\lim f(b_n) = 0$, a contradiction.
Denote $g(x) := x\cos\frac{1}{x}, x \in \mathbb{R} \setminus \{0\}$. Note that $|\cos y| \le 1, \forall y \in \mathbb{R}$. Given $\varepsilon > 0$, set $\delta = \varepsilon$. For any x satisfying $0 < |x| < \delta$,

$$|g(x)| = |x|\,|\cos(1/x)| \le |x| < \varepsilon.$$

Since ε is arbitrary, $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.

4.2-5

By the supposition, there exists $\delta_1 > 0$ and $M > 0$ so that $|f(x)| < M, \forall x \in (c-\delta_1, c+\delta_1)$. Given $\varepsilon > 0$, there exists $\delta_2 > 0$ so that $|g(x)| < \varepsilon/M, \forall x \in (c-\delta_2, c+\delta_2) \setminus \{c\}$. Set $\delta = \min\{\delta_1, \delta_2\}$. For x satisfying $0 < |x-c| < \delta$, we have

$$|f(x)g(x)| < M \cdot (\varepsilon/M) = \varepsilon.$$

4.2-11(c)

Denote $f(x) := \operatorname{sgn}\sin\frac{1}{x}, x \in \mathbb{R}\setminus\{0\}$, $a_n = \frac{1}{2n\pi}, n \in \mathbb{N}$ and $b_n = \frac{1}{(2n+\frac{1}{2})\pi}, n \in \mathbb{N}$. Note that $a_n, b_n \neq 0, \forall n \in \mathbb{N}$ and that (a_n) and (b_n) are convergent sequences with common limit 0. Suppose that $\lim_{x\to 0} f(x) = L$ exist, which implies that $L = \lim(f(a_n)) = \lim(f(b_n))$ by Theorem 4.1.8(b) and Theorem 4.1.5. But $f(a_n) = \operatorname{sgn}\sin 2n\pi = 0$ and $f(b_n) = \operatorname{sgn}\sin(2n+\frac{1}{2})\pi = 1$ for any $n \in \mathbb{N}$. Thus $\lim f(a_n) = 0$ while $\lim f(b_n) = 1$, a contradiction.

5.1-5

For $x \neq 2$,

$$f(x) = \frac{x^2 + x + 6}{x - 2} = \frac{(x-2)(x+3)}{x-2} = x + 3.$$

To hold the continuity at $x = 2$, by the Remark 1 of Theorem 5.1.2, the necessary and sufficient condition is that $f(2) = \lim_{x \to 2} f(x) = 5$. We conclude that $f(x)$ is continuous at $x = 2$ if and only if define $f(2) = 5$.

5.1-8

Given $(x_n) \in S$, denote and fix $x := \lim(x_n)$. Given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ so that $|f(y) - f(x)| < \varepsilon$, for any y satisfying $|y - x| < \delta(\varepsilon)$. For this fixed $\delta(\varepsilon)$, there exist $N \in \mathbb{N}$ so that $|x_n - x| \leq \delta(\varepsilon), \forall n > N$ by the definition of limit. Note that $f(x_n) = 0, \forall n \in \mathbb{N}$. Hence

$$|f(x)| = |f(x) - f(x_{N+1})| < \varepsilon.$$

Since ε is arbitrary, $|f(x)| = 0$ and $f(x) = 0$.