MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 7

3.4-19

Denote $X_k = \sup\{x_n : n \geq k\}, Y_k = \sup\{y_n : n \geq k\}$ and $Z_k = \sup\{x_n + y_n : n \geq k\}$. From the definition of supremum, for all $n \geq k$, $x_n \leq X_k$, $y_n \leq Y_k$ and $x_n + y_n \leq X_k + Y_k$. Thus $X_k + Y_k$ is an upper bound of $\{x_n + y_n, n \geq k\}$. Since Z_k is the supremum of $\{x_n + y_n, n \geq k\}$,

$$Z_k \leq X_k + Y_k, \quad \forall k \in \mathbb{N}.$$

From Theorem 3.4.11(c), $(X_k), (Y_k)$ and (Z_k) are convergent and $\lim(X_k) = \limsup x_n$, $\lim(Y_k) = \limsup y_n$ and $\lim(Z_k) = \limsup(x_n + y_n)$. Theorem 3.2.5 told that for two CONVERGENT sequences (a_n) and (b_n) so that $a_n \leq b_n, \forall n \in \mathbb{N}$, we have $\lim(a_n) \leq \lim(b_n)$. Replacing $a_k = Z_k$ and $b_k = X_k + Y_k$, we have

$$\lim(Z_k) \leq \lim(X_k + Y_k) = \lim(X_k) + \lim(Y_k),$$

i.e.

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

3.5-2(b)

Denote $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ satisfying $\frac{1}{2^{N-1}} < \varepsilon$ by Archimedean Property. If $n \geq N$, $\frac{1}{2^{n-1}} \leq \frac{1}{2^{N-1}} < \varepsilon$. For all $n \geq N$ and $\forall k \in \mathbb{N}$, we have

$$|x_{n+k} - x_n| = \frac{1}{(n+1)!} + \cdots + \frac{1}{(n+k)!} < \frac{1}{2^n} + \cdots + \frac{1}{2^{n+k-1}} < \frac{1}{2^{n-1}} < \varepsilon,$$

which verifies the condition of Cauchy sequence.

3.5-3(b)

Denote $x_n = n + \frac{(-1)^n}{n}$ and $\varepsilon_0 = 2$. For arbitrary $N \in \mathbb{N}$, let $n = N$ and $m = N + 4$. From the triangle inequality $|x + y| \geq ||x| - |y||$,

$$\begin{aligned} |x_{N+4} - x_N| &= \left|4 + (-1)^N(\frac{1}{N+4} - \frac{1}{N})\right| \\ &\geq \left|4 - \left|\frac{1}{N+4} - \frac{1}{N}\right|\right| \\ &= 4 - \left|\frac{1}{N+4} - \frac{1}{N}\right| > 2. \end{aligned}$$

Thus the sequence is not Cauchy.

3.5-9

Given $\varepsilon > 0$, there exist $N \in \mathbb{N}$ so that $r^n < (1 - r)\varepsilon$ for all $n > N$. For all $n > N$ and all $k \in \mathbb{N}$,

$$\begin{aligned} |x_{n+k} - x_n| &= \left|\sum_{i=1}^{k}(x_{n+i} - x_{n+i-1})\right| \\ &\leq \sum_{i=1}^{k} |x_{n+i} - x_{n+i-1}| \\ &< \sum_{i=1}^{k} r^{n+i-1} \\ &\leq \frac{r^n}{1-r} < \varepsilon, \end{aligned}$$

which verifies the condition of Cauchy sequence.

3.5-10

To show the convergence, it suffices to verify that (x_n) is a contractive sequence and apply Theorem 3.5.8. From the iteration formula $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left|\frac{1}{2}(x_{n+1} + x_n) - x_{n+1}\right| \\ &= \left|\frac{1}{2}(x_{n+1} - x_n)\right| \\ &\leq \frac{1}{2}\left|x_{n+1} - x_n\right|, \end{aligned}$$

which verifies the condition of contraction.
To evaluate the limit, by the iteration formula again and for $n \geq 2$,

$$x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2}) = (-\frac{1}{2})^2(x_{n-2} - x_{n-3}) = \cdots = (-\frac{1}{2})^{n-2}(x_2 - x_1).$$

Combining with the identity

$$x_n = x_1 + \sum_{i=2}^{n} (x_i - x_{i-1}),$$

we obtain

$$\begin{aligned} x_n &= x_1 + \sum_{i=2}^{n} (x_i - x_{i-1}) \\ &= x_1 + \sum_{i=2}^{n} (-\frac{1}{2})^{i-2} (x_2 - x_1) \\ &= x_1 + (\frac{1 - (-\frac{1}{2})^{n-1}}{1 - (-\frac{1}{2})})(x_2 - x_1) \\ &= x_1 + \frac{2}{3}(1 - (-\frac{1}{2})^{n-1})(x_2 - x_1). \end{aligned}$$

Thus

$$\lim x_n = \lim[x_1 + \frac{2}{3}(1 - (-\frac{1}{2})^{n-1})(x_2 - x_1)] = x_1 + \frac{2}{3}(x_2 - x_1) = \frac{x_1 + 2x_2}{3}.$$