MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 6

3.3-3

At first, we show that $x_n \geq 2, \forall n \in \mathbb{N}$ by induction. For $k = 1$, $x_1 \geq 2$. Assume that $x_k \geq 2$ for $k = n$. Then $x_{n+1} = 1 + \sqrt{x_n - 1} \geq 1 + \sqrt{2 - 1} \geq 2$. Hence (x_n) is bounded below by 2. Note that $x_n - 1 \geq 1, \forall n \in \mathbb{N}$ and $y \geq \sqrt{y}, \forall y \geq 1$. Thus $x_n - x_{n+1} = x_n - 1 - \sqrt{x_n - 1} \geq 0$. (x_n) is decreasing and bounded above by x_1. Applying Theorem 3.3.2, $\lim x_{n+1} = 1 + \lim \sqrt{x_n - 1}$. Denote $x_0 := \lim x_n$. We have $x_0 - 1 = \sqrt{x_0 - 1}$. By transformation,

$$\begin{aligned} x_0 - 1 = \sqrt{x_0 - 1} &\Leftrightarrow (x_0 - 1)^2 = x_0 - 1 \\ &\Leftrightarrow x_0^2 - 3x_0 - 2 = 0 \\ &\Leftrightarrow (x_0 - 2)(x_0 - 1) = 0. \end{aligned}$$

Since $x_0 = \lim x_n \geq 2$, we have $x_0 = 2$.

3.3-7

Claim: (x_n) always diverges for any $x_1 = a > 0$. At first, we show that $x_n > 0, \forall n \in \mathbb{N}$ by induction. For $k = 1$, $x_1 = a > 0$. Assume that $x_k > 0$ for $k = n$. Then $x_{n+1} = x_n + 1/x_n > 0$. Hence (x_n) is a positive sequence. As $1/x_n > 0$, $x_{n+1} = x_n + 1/x_n > x_n$. (x_n) is increasing. Suppose that (x_n) is convergent and $x_0 := \lim x_n \in \mathbb{R}$. Then $x_0 \geq x_1 > 0$. From $\lim x_{n+1} = \lim x_n + \lim 1/x_n$, we have $0 = \lim 1/x_n = 1/x_0 > 0$, a contradiction. Thus the supposition is false and (x_n) is convergent.

3.3-12(a)

Denote $a_n = (1 + 1/n)^n$ and $b_n = 1 + 1/n$. $(1 + 1/n)^{n+1} = a_n \cdot b_n$. Then we have $\lim a_n = e$ and $\lim b_n = 1$. By Theorem 3.2.3(a), $\lim(1 + 1/n)^{n+1} = \lim a_n \cdot \lim b_n = e$.

3.4-4(a)

To show the sequence is divergent, it suffices to find two convergent subsequences with different limits. Take the subsequence (x_{2n}). $\lim x_{2n} = \lim 1/(2n) = 0$. Also, take the subsequence (x_{2n+1}). $\lim x_{2n+1} = \lim 2 + 1/(2n+1) = 2$. By Theorem 3.4.5, the sequence is divergent.

3.4-9

Since some subsequence of (x_n) converges to 0, the only possible limit of (x_n) is 0. On the contrary, suppose that (x_n) is divergent. Then (x_n) dose not converges to 0. By Theorem 3.4.4(iii), there exists $\epsilon_0 > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - 0| > \epsilon_0$ for all $k \in \mathbb{N}$. By assumption, we can find a further subsequence (which by abuse of notation, we still denote by (x_{n_k}) of (x_{n_k}) converging to 0. So, we have found a subsequence (x_{n_k}) of (x_n) which is converging to 0 and $|x_{n_k}| \geq \varepsilon_0$ for all $k \in \mathbb{N}$ at the same time. This is a contradiction.