MATH 2050C Mathematical Analysis I
2018-19 Term 2

Solution to Problem Set 12

5.4-6

From the assumption, there exists $M > 0$, so that $|f|, |g| < M$ on A. Also, given $\varepsilon > 0$, there exists $\delta > 0$ so that if $x, u \in A$ and $|x - u| < \delta$, then

$$|f(x) - f(u)| < \varepsilon \quad \text{and} \quad |g(x) - g(u)| < \varepsilon.$$

Hence

$$\begin{aligned} |fg(x) - fg(u)| &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\ &\le |f(x)g(x) - f(x)g(u)| + |f(x)g(u) - f(u)g(u)| \\ &\le M\,|g(x) - g(u)| + M\,|f(x) - f(u)| \\ &< 2M\varepsilon. \end{aligned}$$

5.4-12

Since f is uniformly continuous on $[a, \infty)$, given $\varepsilon > 0$, there exists $\delta_1 > 0$ so that if $x, u \in [a, \infty)$ and $|x - u| < \delta_1$, then

$$|f(x) - f(u)| < \varepsilon.$$

Since f is continuous on $[0, a + 1]$, f is uniformly continuous on $[0, a + 1]$ by Theorem 5.4.3. Given $\varepsilon > 0$, there exists $\delta_2 > 0$ so that if $x, u \in [0, a + 1]$ and $|x - u| < \delta_2$, then

$$|f(x) - f(u)| < \varepsilon.$$

Denote $\delta = \min\{\delta_1, \delta_2, 1\}$. Note that either $x, u \in [0, a + 1]$ or $x, u \in [a, \infty)$ for any $x, u \in [0, \infty)$ with $|x - u| < \delta$. Thus $|f(x) - f(u)| < \varepsilon$ in either case.

5.4-14

For any $x \in \mathbb{R}$, $x \in [k_x p, k_x p + p)$ for some $k_x \in \mathbb{Z}$ and $x - k_x p \in [0, p)$, since $\mathbb{R} = \cup_{k \in \mathbb{Z}}[kp, kp + p)$. Denote $M = \sup\{|f(x)|, x \in [0, p]\}$. $M < \infty$ since f is continuous and bounded on $[0, p]$. We have

$$|f(x)| = |f(x - k_x p)| \le M, \quad \forall x \in \mathbb{R},$$

where the periodicity of f is applied. We deduce that f is bounded on $\mathbb{R}$.
To show the uniform continuity, first notice that f is uniformly continuous on $[0, 2p]$. Given $\varepsilon > 0$, there exists $\delta_0(\varepsilon) > 0$ so that if $x, u \in [0, 2p]$ satisfying $|x - u| < \delta_0$, then $|f(x) - f(u)| < \varepsilon$. Now we show the uniform continuity on $\mathbb{R}$. Given $\varepsilon > 0$, denote $\delta = \min\{p, \delta_0(\varepsilon)\}$. Without loss of generality, we assume $x \leq u$. For any $x, u \in \mathbb{R}$ satisfying $|x - u| < \delta$, there are two cases.

(i) $u \in [k_x p, k_x p+p)$. Then $x-k_x p, u-k_x p \in [0, p]$ and $|(x - k_x p) - (u - k_x p)| = |x - u| < \delta$. Thus $|f(x) - f(u)| = |f(x - k_x p) - f(u - k_x p)| < \varepsilon$.

(ii) $u \geq k_x p + p$. Then $u < x + \delta < k_x p + p + p < k_x p + 2p$. We have $x - k_x p, u - k_x p \in [0, 2p]$ and $|(x - k_x p) - (u - k_x p)| = |x - u| < \delta$. Thus $|f(x) - f(u)| = |f(x - k_x p) - f(u - k_x p)| < \varepsilon$.

Combine these two cases and we deduce that f is uniformly continuous on $\mathbb{R}$.

5.6-5

Since f is increasing on I, $f(a) \leq f(x), \forall x \in I$ and $f(a)$ is a lower bound of $\{f(x) : x \in (a, b]\}$. Suppose that $f(a)$ is the infimum of $\{f(x) : x \in (a, b]\}$. From the definition of infimum, given $\varepsilon > 0$, there exists $y_\varepsilon \in (a, b]$ so that $f(y_\varepsilon) < f(a) + \varepsilon$. Denote $\delta(\varepsilon) = y_\varepsilon - a > 0$. For any $y \in [a, b]$ satisfying $|y - a| < \delta(\varepsilon)$, $0 \leq f(y) - f(a) \leq f(y_\varepsilon) - f(a) < \varepsilon$ by the monotonicity of f. Thus f is continuous on a. Conversely, suppose f is continuous on a. Given $\varepsilon > 0$, there exists $\delta > 0$ so that for any $y \in [a, b]$, if $|y - a| < \delta$, $0 \leq f(y) - f(a) < \varepsilon$. Let $y_\varepsilon \in (a, a + \delta) \cap (a, b]$. We have $f(y_\varepsilon) < f(a) + \varepsilon$, which verify the definition of infimum.