THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 8 (March 18)

Definition. Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is said to be a **cluster point** of A if given any $\delta > 0$, there exists $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

Remarks. (1) Equivalently, c is a cluster point of A if and only if

$$V_\delta(c) \cap A \setminus \{c\} \neq \emptyset \quad \text{for any } \delta > 0.$$

(2) A cluster point of A may or may not be an element of A.

Definition. Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

In this case, we write

$$\lim_{x\to c} f = L, \quad \lim_{x\to c} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to c.$$

Theorem. *If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Example 1. Use the ε-δ definition of limit to show that $\lim_{x\to -3}(x^2 + 4x) = -3$.

Solution. Observe that $x^2 + 4x$ has a natural domain $\mathbb{R}$, which clearly has -3 as a cluster point.

Note that $|(x^2 + 4x) - (-3)| = |x^2 + 4x + 3| = |x + 1||x + 3|$.

If $|x + 3| < 1$, then $|x + 1| = |(x + 3) - 2| \leq |x + 3| + 2 < 3$.

Let $\varepsilon > 0$ be given. Take $\delta := \min\{\varepsilon/3, 1\}$. Now if $0 < |x - (-3)| < \delta$, then

$$|(x^2 + 4x) - (-3)| = |x + 1||x + 3| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\lim_{x\to -3}(x^2 + 4x) = -3$. ◀

Example 2. Use the ε-δ definition of limit to show that $\lim_{x\to 2} \frac{x + 6}{x^2 - 2} = 4$.

Solution. Clearly $f(x) := \frac{x + 6}{x^2 - 2}$ has a natural domain $\mathbb{R} \setminus \{\pm\sqrt{2}\}$, which has 2 as a cluster point.

For $x \in \mathbb{R} \setminus \{\pm\sqrt{2}\}$,

$$|f(x) - 4| = \left|\frac{x + 6}{x^2 - 2} - 4\right| = \frac{|4x^2 - x - 14|}{|x^2 - 2|} = \frac{|4x + 7|}{|x^2 - 2|} \cdot |x - 2|.$$

If $|x-2| < \dfrac{1}{2}$, then

$$\frac{3}{2} < x < \frac{5}{2} \implies \frac{1}{4} < x^2 - 2 < \frac{17}{4},$$

and

$$|4x+7| = |4(x-2)+15| \le 4|x-2| + 15 \le 20.$$

Let $\varepsilon > 0$ be given. Take $\delta := \min\left\{\dfrac{\varepsilon}{80}, \dfrac{1}{2}\right\}$. Now if $0 < |x-2| < \delta$, then

$$|f(x) - 4| = \frac{|4x+7|}{|x^2-2|} \cdot |x-2| < \frac{20}{1/4} \cdot \frac{\varepsilon}{80} = \varepsilon.$$

Hence $\displaystyle\lim_{x\to 2} \frac{x+6}{x^2-2} = 4$. ◀

Example 3. Use the ε-δ definition of limit to evaluate the limit $\displaystyle\lim_{x\to 4} \frac{4-x}{2-\sqrt{|x|}}$.

Solution. For $x \in [0,\infty)\backslash\{4\}$,

$$\frac{4-x}{2-\sqrt{|x|}} = \frac{4-x}{2-\sqrt{x}} = \frac{(2+\sqrt{x})(2-\sqrt{x})}{2+\sqrt{x}} = 2 + \sqrt{x},$$

and thus

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| = |\sqrt{x} - 2| = \left|\frac{x-4}{\sqrt{x}+2}\right| \le \frac{|x-4|}{2}.$$

Let $\varepsilon > 0$ be given. Set $\delta = \min\{\varepsilon, 1\}$. Now, if $0 < |x-4| < \delta$, then $x \in (3,5)\backslash\{4\}$, and hence

$$\left|\frac{4-x}{2-\sqrt{|x|}} - 4\right| \le \frac{|x-4|}{2} < \frac{\delta}{2} \le \varepsilon.$$

Therefore

$$\lim_{x\to 4} \frac{4-x}{2-\sqrt{|x|}} = 4.$$

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Classwork

Use the ε-δ definition of limit to evaluate the following limits.

(a) $\displaystyle\lim_{x\to 3} f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} x-1 & \text{if } x \in \mathbb{R} \text{ is rational} \\ 8-2x & \text{if } x \in \mathbb{R} \text{ is irrational.} \end{cases}$$

(b) $\displaystyle\lim_{x\to 2} \frac{3x-4}{x^2-3} = 2$.

Solution. (a) For any x,

$$\begin{aligned} |f(x) - 2| &= \begin{cases} |x - 1 - 2| & \text{if } x \in \mathbb{R} \text{ is rational} \\ |8 - 2x - 2| & \text{if } x \in \mathbb{R} \text{ is irrational} \end{cases} \\ &\leq 2|x - 3|. \end{aligned}$$

Let $\varepsilon > 0$ be given. Set $\delta = \varepsilon/2$. If $0 < |x - 3| < \delta$, then

$$|f(x) - 2| \leq 2|x - 3| < 2\delta = \varepsilon.$$

Therefore $\lim_{x \to 3} f(x) = 2$.

(b) For $x \neq \pm\sqrt{3}$,

$$\left| \frac{3x - 4}{x^2 - 3} - 2 \right| = \frac{|2x + 1|}{|x^2 - 3|} |x - 2|.$$

If $|x - 2| < 1/4$, then

$$\frac{7}{4} < x < \frac{9}{4} \implies \frac{1}{16} < x^2 - 3 < \frac{33}{16},$$

and

$$|2x + 1| \leq |2(x - 2) + 5| \leq 2|x - 2| + 5 < 10.$$

Let $\varepsilon > 0$. Take $\delta = \min\{\varepsilon/160, 1/4\}$. Now if $0 < |x - 2| < \delta$, then

$$\left| \frac{3x - 4}{x^2 - 3} - 2 \right| = \frac{|2x + 1|}{|x^2 - 3|} |x - 2| < \frac{10}{1/16} \cdot \frac{\varepsilon}{160} < \varepsilon.$$

Hence $\lim_{x \to 2} \frac{3x - 4}{x^2 - 3} = 2$.

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