THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 5 (February 25)

Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent if and only if it is bounded. Furthermore,*

(a) If (x_n) is a bounded increasing sequence, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.

(b) If (y_n) is a bounded decreasing sequence, then $\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$.

Example 1. Let $Z = (z_n)$ be the sequence of real numbers defined by

$$z_1 \coloneqq 1, \quad z_{n+1} \coloneqq \sqrt{2z_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $\lim(z_n) = 2$.

Example 2 (*Euler number e*)**.** Let $e_n \coloneqq (1 + 1/n)^n$ for $n \in \mathbb{N}$. Show that the sequence $E = (e_n)$ is bounded and increasing, hence convergent. The limit of this sequence is called the *Euler number*, and it is denoted by e.

Example 3. Establish the convergence and find the limits of the following sequences.

(a) $((1 + 1/n)^{n+1})$

(b) $\left(\left(1 + \dfrac{1}{n+1}\right)^n\right)$

(c) $((1 - 1/n)^n)$

Classwork

1. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. (Hint: $1 + 2\sqrt{p}$ is one upper bound.)

   Solution. Note $y_2 = \sqrt{p + \sqrt{p}} > \sqrt{p} = y_1$. Suppose $y_{k+1} > y_k$ for some $k \in \mathbb{N}$. Then
   $$y_{k+2} = \sqrt{p + y_{k+1}} > \sqrt{p + y_k} = y_{k+1}.$$
   By induction, $y_{n+1} > y_n$ for all $n \in \mathbb{N}$.

   Note $y_1 = \sqrt{p} < 1 + 2\sqrt{p}$. Suppose $y_k < 1 + 2\sqrt{p}$ for some $k \in \mathbb{N}$. Then
   $$y_{k+1} = \sqrt{p + y_k} < \sqrt{p + 1 + 2\sqrt{p}} = \sqrt{(1 + \sqrt{p})^2} < 1 + 2\sqrt{p}.$$
   By induction, $y_n < 1 + 2\sqrt{p}$. for all $n \in \mathbb{N}$.

   The sequence (y_n) is thus increasing and bounded above. By Monotone Convergence Theorem, $y := \lim(y_n)$ exists. Since $y_{n+1} = \sqrt{p + y_n}$, we have
   $$y = \sqrt{p + y} \implies y^2 - y - p = 0 \implies y = \frac{1}{2}\left(1 \pm \sqrt{1 + 4p}\right).$$
   Since $y_n > 0$ for all $n \in \mathbb{N}$, we have $y \geq 0$ and hence $y = \frac{1}{2}\left(1 + \sqrt{1 + 4p}\right)$. ◀

2. Let $b_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$ for $n \in \mathbb{N}$. Show that (b_n) is convergent. Furthermore, show that
   $$\lim(b_n) = \lim(e_n) = e.$$

   Solution. It is easy to see that (b_n) is increasing. In Example 2, it is shown that $e_n < b_n < 3$ for $n \in \mathbb{N}$. Hence, by Monotone Convergence Theorem, (b_n) converges. Let $\ell = \lim(b_n)$. Then $e \leq \ell$. On the other hand, fix $N \in \mathbb{N}$. For $n \geq N$, we have
   $$e_n \geq 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{N!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{N-1}{n}\right).$$
   Passing $n \to \infty$, we get $e \geq b_N$. Since N is arbitrary, it implies that $e \geq \ell$. Therefore $e = \ell$. ◀