

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050C Mathematical Analysis I
Tutorial 3 (February 4)

1 The Limit of a Sequence

Definition. A sequence $X = (x_n)$ in \mathbb{R} is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n) , if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

Notations: $\lim X = x$, $\lim(x_n) = x$, $\lim_n x_n = x$, $\lim_{n \rightarrow \infty} x_n = x$.

Procedure. To show that $\lim(x_n) = x$, we proceed as follow:

- (1) Fix an $\varepsilon > 0$. (ε is arbitrary, but cannot be changed once fixed.)
- (2) Find a useful estimate for $|x_n - x|$.
- (3) Find $K(\varepsilon) \in \mathbb{N}$ such that the estimate in (2) is less than ε whenever $n \geq K(\varepsilon)$.
- (4) Complete the proof.

Example 1. Use the definition to show that $\lim \frac{1}{n^2 + 1} = 0$.

Solution. Let $\varepsilon > 0$ be given. Note that

$$\left| \frac{1}{n^2 + 1} - 0 \right| = \frac{1}{n^2 + 1} < \frac{1}{n^2} \leq \frac{1}{n} \quad \text{for } n \in \mathbb{N}.$$

By Archimedean Property, there is $K \in \mathbb{N}$ such that $K > 1/\varepsilon$. Now if $n \geq K$, then $1/n \leq 1/K < \varepsilon$, and thus

$$\left| \frac{1}{n^2 + 1} - 0 \right| \leq \frac{1}{n} < \varepsilon.$$

Hence $\lim 1/(n^2 + 1) = 0$. ◀

Example 2. Use the definition to show that $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.

Solution. We multiply and divide by $\sqrt{n+1} + \sqrt{n}$ to get

$$0 < \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1})^2 - (\sqrt{n})^2}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}.$$

Let $\varepsilon > 0$. By Archimedean Property, there is $K \in \mathbb{N}$ such that $K > 1/\varepsilon^2$. Now if $n \geq K$, we have $1/\sqrt{n} \leq 1/\sqrt{K} < \varepsilon$, and hence

$$\left| \sqrt{n+1} - \sqrt{n} \right| < \frac{1}{\sqrt{n}} < \varepsilon. \quad \text{◀}$$

Definition. If $X = (x_1, x_2, x_3, \dots, x_n, \dots)$ is a sequence of real numbers and if m is a given natural number, then the m -**tail** of X is the sequence

$$X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \dots).$$

For example, if $X = (1/n : n \in \mathbb{N})$, then $X_{1997} = (1/1998, 1/1999, \dots)$.

Theorem. Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m -tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case, $\lim X_m = \lim X$.

Proof. Write $X_m = (y_k : k \in \mathbb{N})$. Then $y_k = x_{k+m}$ for any $k \in \mathbb{N}$.

Assume X converges to x . Then given any $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ with $K(\varepsilon) > m$ such that

$$|x_k - x| < \varepsilon \quad \text{for all } k \geq K(\varepsilon),$$

which implies that

$$|y_k - x| = |x_{k+m} - x| < \varepsilon \quad \text{for all } k \geq K(\varepsilon) - m.$$

By taking $K_m(\varepsilon) = K(\varepsilon) - m$, we conclude that X_m converges to x .

Conversely, assume that X_m converges to x . Then given any $\varepsilon > 0$, there is $K_m(\varepsilon) \in \mathbb{N}$ such that

$$|y_k - x| < \varepsilon \quad \text{for all } k \geq K_m(\varepsilon),$$

which implies that

$$|x_k - x| = |y_{k-m} - x| < \varepsilon \quad \text{for all } k \geq K_m(\varepsilon) + m.$$

By taking $K(\varepsilon) = K_m(\varepsilon) + m$, we conclude that X converges to x .

Therefore, X converges to x if and only if X_m converges to x .

□

Classwork

1. Use the definition to show that $\lim \left(\frac{n^2 - n}{2n^2 + 3} \right) = \frac{1}{2}$.
2. If $\lim(x_n) = x$ and $x \neq 0$, show that there exists a natural number K such that if $n \geq K$, then $\frac{1}{2}|x| < |x_n| < 2|x|$.