THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 2 (January 28)

1 The Completeness Property of $\mathbb{R}$

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 1. (a) Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any real number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. Show that

$$\sup(a + S) = a + \sup S.$$

(b) Let A and B be nonempty subsets of $\mathbb{R}$ that satisfy the property:

$$a \leq b \qquad \text{for all } a \in A \text{ and } b \in B.$$

Show that $\sup A \leq \inf B$.

Example 2. Suppose that f and g are real-valued functions with common domain $D \subseteq \mathbb{R}$. Assume that f and g are bounded (that is, $f(D)$ and $g(D)$ are bounded subsets of $\mathbb{R}$).

(a) If $f(x) \leq g(x)$ for all $x \in D$, show that $\sup f(D) \leq \sup g(D)$.

(b) If $f(x) \leq g(x)$ for all $x \in D$, is it true that $\sup f(D) \leq \inf g(D)$?

(c) If $f(x) \leq g(y)$ for all $x, y \in D$, show that $\sup f(D) \leq \inf g(D)$.

Solution. (c) Given $y \in D$, we have $f(x) \leq g(y)$ for all $x \in D$. So $g(y)$ is an upper bound for $f(D)$. Hence $\sup f(D) \leq g(y)$. Since the last inequality holds for all $y \in D$, we see that $\sup f(D)$ is a lower bound for $g(D)$. Therefore, we conclude that $\sup f(D) \leq \inf g(D)$. ◀

Classwork

1. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that if $b < 0$,

$$\inf(bS) = b \sup S.$$

2. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B.$$

Solution of Classwork

1. Let S be a nonempty bounded subset of $\mathbb{R}$. Show that if $b < 0$,

$$\inf(bS) = b \sup S.$$

Solution. By the completeness property, $\sup S$ exists.

For any $s \in S$, we have $s \leq \sup S$, so that $bs \geq b \sup S$ since $b < 0$. Hence $b \sup S$ is a lower bound of bS.

Suppose $v > b \sup S$. Then $v/b < \sup S$ since $b < 0$. So there exists $s_v \in S$ such that $v/b < s_v$, which implies that $v > bs_v$. Hence $b \sup S$ is the greatest lower bound of bS, that is $\inf(bS) = b \sup S$. ◀

2. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B \quad \text{and} \quad \inf(A + B) = \inf A + \inf B.$$

Solution. We only prove the first one.

By the completeness property, $\sup A$ and $\sup B$ both exist.

For $a \in A$, $b \in B$, we have $a \leq \sup A$, $b \leq \sup B$, so that

$$a + b \leq \sup A + \sup B.$$

Hence $A + B$ is bounded above by $\sup A + \sup B$. By the completeness property, $\sup(A + B)$ exists and

$$\sup(A + B) \leq \sup A + \sup B.$$

On the other hand, fix $b \in B$. Then, for $a \in A$,

$$a + b \leq \sup(A + B) \implies a \leq \sup(A + B) - b.$$

Hence RHS is an upper bounded of A, and thus

$$\sup A \leq \sup(A + B) - b \implies b \leq \sup(A + B) - \sup A. \tag{1}$$

Since (1) is true for any $b \in B$, RHS is an upper bound of B, and thus

$$\sup B \leq \sup(A + B) - \sup A,$$

that is

$$\sup(A + B) \geq \sup A + \sup B.$$

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