THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 12 (April 22)

Boundedness Theorem. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I. Then f is bounded on I.*

Maximum-Minimum Theorem. *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function on I. Then f has an absolute maximum and an absolute minimum on I, that is, there exist $x_*, x^* \in I$ such that*

$$f(x_*) \leq f(x) \leq f(x^*) \qquad \text{for all } x \in I.$$

Example 1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $\lim_{x\to-\infty} f = 0$ and $\lim_{x\to\infty} f = 0$.

(a) Prove that f is bounded on $\mathbb{R}$.

(b) Prove that f attains either a maximum or minimum on $\mathbb{R}$.

(c) Give an example to show that both a maximum and a minimum need not be attained.

Uniform Continuity Theorem. *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

Example 2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic on $\mathbb{R}$ if there exists a number $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function on $\mathbb{R}$ with $f(0) = 0$. Prove that there exists some $C > 0$ such that

$$|f(x)| \leq 1 + C|x| \qquad \text{for all } x \in \mathbb{R}.$$