THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2050C Mathematical Analysis I

Tutorial 1 (January 21)

1 Order Properties of $\mathbb{R}$

The Order Properties of $\mathbb{R}$. *There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:*

(I) $a, b \in \mathbb{P} \implies a + b \in \mathbb{P}$,

(II) $a, b \in \mathbb{P} \implies ab \in \mathbb{P}$,

(III) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

Write $a > 0$ if $a \in \mathbb{P}$; and write $a > b$ if $a - b \in \mathbb{P}$.

Example 1. If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n > 1$.

Solution. Clearly $c^1 = c$. We will prove that $c^n < c$ for $n \geq 2$ by induction.

Since $0 < c < 1$, we have $c \cdot c < c \cdot 1$ by Theorem 2.1.7(c). Hence $c^2 < c$.

Assume the validity of the inequality for some $k \in \mathbb{N}$. Then Theorem 2.1.7(c) again implies that

$$c^{k+1} = c \cdot c^k < c \cdot c = c^2 < c.$$

Thus, the inequality holds for $n = k + 1$.

By induction, $c^n < c$ for $n \geq 2$. ◀

Triangle Inequality. *If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$.*

Example 2 (Reverse Triangle Inequality)**.** If $a, b \in \mathbb{R}$, show that $||a| - |b|| \leq |a - b|$.

Solution. Write $a = (a - b) + b$. By the Triangle Inequality, we have

$$|a| = |(a - b) + b| \leq |a - b| + |b|.$$

Subtract $|b|$ to get $|a| - |b| \leq |a - b|$. Similarly,

$$|b| = |(b - a) + a| \leq |b - a| + |a|,$$

so that

$$-|a - b| = -|b - a| \leq |a| - |b|.$$

Combining the two inequalities and using Theorem 2.2.2(c), we get $||a| - |b|| \leq |a - b|$. ◀

2 The Completeness Property of $\mathbb{R}$

Definition. Let S be a nonempty subset of $\mathbb{R}$. Suppose S is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if it satisfies the conditions:

(i) u is an upper bound of S (that is, $s \le u$ for all $s \in S$), and

(ii) if v is any upper bound of S, then $u \le v$.

Here (ii) is equivalent to

(ii)' if $v < u$, then there exists $s_v \in S$ such that $v < s_v$.

Remarks. (1) The number u is unique and we write $\sup S = u$.

(2) u may or may not be an element of S.

(3) $\inf S$ can be defined similarly provided S is bounded below.

Example 3. Let $S := \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S$ and $\sup S$.

Solution. Since

$$-1 \le (-1)^n/n \le 1/2 \quad \text{for } n \in \mathbb{N},$$

we have

$$1/2 = 1 - 1/2 \le 1 - (-1)^n/n \le 1 - (-1) = 2 \quad \text{for } n \in \mathbb{N}.$$

Hence S is bounded above by 2 and bounded below by 1/2.

First we show that $\sup S = 2$. Suppose v is an upper bound of S. Then $1 - (-1)^n/n \le v$ for any $n \in \mathbb{N}$. In particular, by taking $n = 1$, we have $1 - (-1)^1/1 = 2 \le v$. Thus 2 is the least upper bound of S, that is, $\sup S = 2$.

Next we show that $\inf S = 1/2$. Suppose $w > 1/2$. Then w is not a lower bound of S since $1/2 = 1 - (-1)^2/2 \in S$ but $1/2 < w$. Thus 1/2 is the greatest lower bound of S, that is, $\inf S = 1/2$. ◀

Classwork

1. Let $A := \{x \in \mathbb{R} : |x + 2| > |x - 1| - 1\}$.
   (a) What are the elements of the set A?
   (b) Is A bounded above? Is A bounded below?
   (c) Find $\sup A$ and $\inf A$, if they exist. Justify your answer.

2. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that

$$\inf\{-s : s \in S\} = -\sup S.$$

Solution of Classwork

1. Let $A := \{x \in \mathbb{R} : |x+2| > |x-1| - 1\}$.

   (a) What are the elements of the set A?

   (b) Is A bounded above? Is A bounded below?

   (c) Find $\sup A$ and $\inf A$, if they exist. Justify your answer.

   Solution. (a) $A = (-1, \infty)$.

   (b) A is not bounded above but bounded below by -1.

   (c) $\sup A$ does not exist since A is not bounded above.
   Let $v > -1$. Take $s_v := (v-1)/2$. Then $s_v > -1 \implies s_v \in A$. Moreover, $s_v < v$. So any $v > -1$ is not a lower bound of S. Hence $\inf A = -1$.

   ◀

2. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that

$$\inf\{-s : s \in S\} = -\sup S.$$

   (Note that $\sup S$ exists by the completeness property of $\mathbb{R}$.)

   Solution. Let $-S := \{-s : s \in S\}$. For any $s \in S$, we have $s \leq \sup S$, so that $-\sup S \leq -s$. Hence $-\sup S$ is a lower bound of $-S$.

   If v is any lower bound of $-S$, then $v \leq -s$ for any $s \in S$. Then $s \leq -v$ for any $s \in S$, and hence $-v$ is an upper bound of S. By the definition of supremum, we have $\sup S \leq -v$, so that $v \leq -\sup S$. Therefore $-\sup S$ is the greatest lower bound of $-S$, that is $\inf(-S) = -\sup S$. ◀