Homework 4 Solution

1) This problem is done by a direct calculation.

2) a) This calculation is most easily done at a point P in a coordinate chart $x^1, \ldots, x^m$ centered at P for which $g_{ij}(0) = \delta_{ij}$ and $\Gamma^i_{jk}(0) = 0$ (for example, Riemannian normal coordinates provide such a chart). We then have

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + u_{,k}\delta^i_j + u_{,j}\delta^i_k - g^{ip}u_{,p}g_{jk}.$$

Therefore we have by direct calculation

$$\hat{R}_{ijkl} = e^{2u}R_{ijkl} + \mathbf{h}_{ijkl}$$

where $\mathbf{h}_{ijkl} = h_{ik}g_{jl} - h_{il}g_{jk} + h_{jl}g_{ik} - h_{jk}g_{il}$ and

$$h_{ij} = e^{2u}(-u_{;ij} + u_{;i}u_{;j} - \frac{1}{2}|\nabla u|^2 g_{ij}).$$

where $|\nabla u|^2 = g^{pq}u_{;p}u_{;q}$. It follows that

$$\hat{R}_{ik} = R_{ik} + (m-2)h_{ik} + Tr_g(h)g_{ik}$$

and

$$\hat{R} = e^{-2u}\big(R + (2m-2)Tr_g(h)\big) = e^{-2u}R - (2m-2)(\Delta u + \frac{m-2}{2}|\nabla u|^2)$$

where $\Delta u = g^{pq}u_{;pq}$.

b) From the form of the transformation of the Riemann curvature tensor in part a we see that $\mathbf{W}(\hat{g}) = e^{2u}\mathbf{W}(g)$.

3) a) Let e_1, e_2, e_3 be an orthonormal basis, and let W_{ijkl} be the components of W in this basis. First observe that all components W_{ijkl} vanish if three of i, j, k, l are distinct. For example, if $i = k$ and i, j, l are distinct, then $W_{ijil} = \sum_{p=1}^{3} W_{pjpl} = 0$. Now if we fix l and consider the symmetric quadratic form $h_{ij} = W_{iljl}$, then we see that $h_{ij} = \lambda_i\delta_{ij}$ in any orthonormal basis for the orthogonal complement of e_l. It follows that $h_{ij} = \lambda\delta_{ij}$ for a fixed number λ. Taking the trace we find $W_{plpl} = 2\lambda = 0$, and therefore $\lambda = 0$. Thus we have $W_{ijkl} = 0$ whenever two of i, j, k, l are distinct and thus all components of W are zero and thus $W = 0$.

If g is an Einstein metric on a three manifold we then have $W = 0$ and $Ric_0 = 0$, so $\mathbf{Riem} = \mathbf{R}$ and R is constant. Therefore g is a constant curvature metric.

b) We may check the Einstein condition in any orthonormal basis, so we choose e_1, e_2 to be tangent to the first factor of S^2 and e_3, e_4 tangent to the second factor. We then have $R_{1j1l} = \delta_{2j}\delta_{2l}$, $R_{2j2l} = \delta_{1j}\delta_{1l}$, $R_{3j3l} = \delta_{4j}\delta_{4l}$, and $R_{4j4l} = \delta_{3j}\delta_{3l}$. It follows that $\sum_{p=1}^{4} R_{pjpl} = \delta_{jl}$, and g is Einstein.

4) a) Clearly $Hf(\varphi X, Y) = \varphi Hf(X, Y)$, and $Hf(X, \varphi Y) = \varphi XYf + (X\varphi)(Yf) - \varphi D_X Y(f) - (X\varphi)(Yf) = \varphi Hf(X, Y)$. Therefore Hf is a tensor. We have $Hf(X, Y) - Hf(Y, X) = [X, Y]f - D_X Y + D_Y X = T(X, Y)(f)$. Thus if D is torsion free, then Hf is symmetric. Conversely, if Hf is symmetric then $T(X, Y)$ applied to any function is 0, and therefore $T(X, Y) = 0$.

b) In coordinates we have

$$Hf(\partial/\partial x^i, \partial/\partial x^j) = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}.$$

5) From problem 4b we have

$$\Delta f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x^k}\right).$$

To verify the other expression, we have

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j}\right) = g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ik}) \frac{\partial f}{\partial x^k},$$

so we must show that $\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ik}) = -g^{ij}\Gamma_{ij}^k$. To verify this we compute

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ik}) = \frac{\partial \log \sqrt{g}}{\partial x^i} g^{ik} + \frac{\partial g^{ik}}{\partial x^i} = \frac{1}{2} g^{pq} g_{pq.i} g^{ik} - g^{ip} g^{kq} g_{pq.i}.$$

Renaming the summation variables this becomes

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ik}) = -g^{ij} g^{kl}\big(g_{il,j} - \frac{1}{2} g_{ij,l}\big) = -g^{ij}\Gamma_{ij}^k,$$

completing the proof.

b) Recall that the hyperbolic metric on $\mathbb{R}_+^m$ is given by $g_{ij} = (x^m)^{-2}\delta_{ij}$, so from part a we have $\Delta_g f = (x^m)^m \frac{\partial}{\partial x^i}((x^m)^{2-m} \frac{\partial f}{\partial x^i})$. Setting $f = (x^m)^p$ we find $\Delta_g f = p(1 - m + p)f$.

c) For $m = 2$ we have $\Delta_g f = (x^2)^2 \sum_i \Delta_0 f$ where Δ_0 is the euclidean Laplacian $\Delta_0 f = \sum_i \frac{\partial^2 f}{\partial (x^i)^2}$, and so the hyperbolic harmonic functions are precisely the euclidean harmonic functions.