Homework 2 Solutions

1) a) We may assume by the Whitney embedding theorem that M is embedded in $\mathbb{R}^n$. A vector field X on M can be extended to a smooth vector field on $\mathbb{R}^n$. If $x_0 \in M$, then the integral curve $\gamma(t)$ with $\gamma(0) = x_0$ is contained in M and therefore remains in a compact subset of $\mathbb{R}^n$. Assume that $(-t_0, t_1)$ is the maximal interval of existence. The map $\gamma : (-t_0, t_1) \to \mathbb{R}^n$ is Lipschitz since $\|\gamma'(t)\| = \|X(\gamma(t))\| \le C$ and hence uniformly continuous. It follows that γ extends continuously to the closed interval. Therefore, if $t_1 < \infty$ we can extend γ beyond t_1. It follows that $t_1 = \infty$ and similarly $t_0 = \infty$. Therefore X is complete.
b) Let x be the standard coordinate on $\mathbb{R}$, and consider the vector field $X = x^2\partial/\partial x$. The associated ODE is $dx/dt = x^2$, and the solution of this with $x(0) = 1$ is $x(t) = 1/(1-t)$ which only exists for $t < 1$.

2) a) Assume that $X = \partial/\partial x^1$. The flow is then given by $\varphi_t(x) = (x^1 + t, x^2, \ldots, x^m)$ and we have $\varphi_{-t*}(\partial/\partial x^i) = \partial/\partial x^i$ for $i = 1, \ldots, m$. If we write $Y = \sum_{i=1}^m b^i(x)\partial/\partial x^i$, then we have

$$\big((\varphi_{-t})_*(Y(\varphi_t(P))-Y(P)\big)/t = \sum_{i=1}^m \big((b^i(x^1+t, x^2, \ldots, x^m)-b^i(x))/t\big)\partial/\partial x^i.$$

Letting $t \to 0$ we see that $L_XY = \sum_{i=1}^m \frac{\partial b^i}{\partial x^1}\frac{\partial}{\partial x^i}$. Direct calculation then shows that this is also the expression for $[X, Y]$ in this coordinate system.
b) From their definition both L_XY and $[X, Y]$ are well defined vector fields on M, and thus to check that they coincide it is sufficient to check this in any coordinate system.

If $P \in M$ is such that $X(P) = 0$, then we consider two cases. If there is a sequence of points $P_i \to P$ such that $X(P_i) \neq 0$, then we have $L_XY = [X, Y]$ at P_i and since both sides are smooth vector fields (hence continuous) it follows that $L_XY = [X, Y]$ at P. The other possibility is that there is a neighborhood U of P in which X is zero. In this case we have $\varphi_t(Q) = Q$ for $Q \in U$, and therefore we see directly that $L_XY = 0$ at P. Similarly $[X, Y] = 0$ at P. Thus in either case we have $L_XY = [X, Y]$ at P.

3) a) Let $\sigma \in \wedge^{m-1}(V)$ and consider the linear transformation $L : V \to \wedge^m(V)$ given by $L(v) = \sigma \wedge v$. If $\sigma \neq 0$, then L is nonzero (see Problem 4a). Since $\wedge^m(V)$ is 1 dimensional, it follows that the nullspace of L is $m-1$ dimensional. Let $v_1, \ldots, v_{m-1}$ be a basis for the nullspace, and complete it to a basis of V by adding an additional vector v_m. The

expression of σ in this basis is then of the form $av_1 \wedge \ldots \wedge v_{m-1}$ since any term of the form $v_{i_1} \wedge \ldots \wedge v_{i_{m-2}} \wedge v_m$ with $1 \leq i_1 < \ldots < i_{m-2} \leq m-1$ has nonzero wedge product with v_j for some j with $1 \leq j \leq m-1$. Therefore σ is simple.

b) Let $e_1, \ldots e_4$ be the standard basis for $\mathbb{R}^4$ and let $\sigma = e_1 \wedge e_2 + e_3 \wedge e_4$. We see that $\sigma \wedge \sigma = 2e_1 \wedge \ldots \wedge e_4 \neq 0$ and therefore σ cannot be simple.

c) Another basis $v_1, \ldots, v_k$ for W would be of the form $v_j = \sum_{i=1}^k a_j^i e_i$ where $A = (a_j^i)$ is a nonsingular $k \times k$ matrix. We then have $v_1 \wedge \ldots \wedge v_k = \det(A) e_1 \wedge \ldots \wedge e_k$. It follows that the map F from k-dimensional subspaces of V to simple elements of the projective space $P = P(\wedge^k(V))$ given by $F(W) = [e_1 \wedge \ldots \wedge e_k]$ is well defined (we use the notation $[\sigma]$ to denote the line through the origin containing a nonzero element of $\wedge^k(V)$). It is clear that the map F is onto. To see that it is one-one, suppose that $F(W_1) = F(W_2)$. Let $e_1, \ldots, e_k$ be a basis for W_1 and $v_1, \ldots, v_k$ a basis for W_2. We then have $v_1 \wedge \ldots \wedge v_k = ae_1 \wedge \ldots \wedge e_k$ for a nonzero number a. By completing $e_1, \ldots, e_k$ to a basis for V and expressing a vector v in terms of this basis, we can see that v is in W_1 if and only if $v \wedge e_1 \wedge \ldots \wedge e_k = 0$. It follows that $v_i \in W_1$ for $i = 1, \ldots, k$ and thus $W_2 = W_1$ as required.

4) a) It suffices to show that if $\alpha \in \wedge^k(V)$ such that $\alpha \wedge \beta = 0$ for all $\beta \in \wedge^{m-k}(V)$ then $\alpha = 0$. To see this we express α in terms of a basis and show that each of the coefficients is 0. To show that the coefficient of the monomial $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is zero, we wedge α with the complementing monomial $\beta = e_{j_1} \wedge \ldots \wedge e_{j_{m-k}}$. Since the wedge product of β with $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is nonzero while the wedge product with all other basis elements of $\wedge^k(V)$ is zero, the condition that $\alpha \wedge \beta = 0$ implies that the coefficient of α corresponding to the monomial $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is zero. Since this was an arbitrary basis element, it follows that $\alpha = 0$.

b) We define $*\beta$ to be the unique element of $\wedge^{m-k}(V)$ which satisfies $\alpha \wedge *\beta = \langle \alpha, \beta \rangle * 1$ for all $\alpha \in \wedge^k(V)$. By part a there is a unique such element and $*$ defines a linear transformation from $\wedge^k(V)$ to $\wedge^{m-k}(V)$. since these vector spaces are of the same dimension, to check that $*$ is an isomorphism it suffices to check that $*\beta = 0$ only if $\beta = 0$. To see this, note that if $*\beta = 0$, then we have $\langle \alpha, \beta \rangle = 0$ for all $\alpha \in \wedge^k(V)$ and hence $\beta = 0$ since g is nondegenerate.

c) We may replace $v_1, \ldots, v_k$ by orthonormal vectors and complete to an orthonormal basis $v_1, \ldots, v_m$ of V. If we take any basis element $\alpha = v_{i_1} \wedge \ldots \wedge v_{i_k}$ of $\wedge^k(V)$, and we let $\sigma = v_{k+1} \wedge \ldots \wedge v_m$, then we have $\alpha \wedge \sigma = 0$ unless $\alpha = v_1 \wedge \ldots \wedge v_k$, and $v_1 \wedge \ldots \wedge v_k \wedge \sigma = *1$.

Therefore $\sigma = *(v_1 \wedge \ldots \wedge v_k)$ which represents the orthogonal $(m-k)$-plane.

d) From part c we see that $*$ takes an orthonormal basis to an orthonormal basis and therefore is an isometry from $\wedge^k(V)$ to $\wedge^{m-k}(V)$. Thus we have for $\alpha, \beta \in \wedge^k(V)$, $*\alpha \wedge *^2\beta = \langle *\alpha, *\beta \rangle * 1 = \alpha \wedge *\beta$. Now $\alpha \wedge *\beta = \beta \wedge *\alpha = (-1)^{k(m-k)} * \alpha \wedge \beta$, so we have $*\alpha \wedge *^2\beta = (-1)^{k(m-k)} * \alpha \wedge \beta$. Since this holds for all α, $*$ is a linear isomorphism, and the wedge product pairing is nondegenerate, we can conclude that $*^2\beta = (-1)^{k(m-k)}\beta$ for all $\beta \in \wedge^k(V)$ as required.

5) a) Let e_1, e_2 be an orthonormal basis with $*1 = e_1 \wedge e_2$. We then have $*e_1 = e_2$ and $*e_2 = -e_1$, so this is the linear transformation which rotates by 90^o in the counterclockwise direction (as defined by the orientation).

b) We have $*^2 = 1$ by Problem 4d, and so the eigenvalues of $*$ are 1 and -1. If we choose an orthonormal basis $e_1, \ldots, e_4$ with positive orientation, then we can explicitly write bases for the eigenspaces. We have

$$\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\}$$

is a basis for the $+1$ eigenspace, and

$$\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\}$$

for the -1 eigenspace.