Homework 1 Solutions

1) Denote by M the smooth manifold which is given by an atlas on the topological manifold $\mathbb{R}$. In order to show that M is diffeomorphic to $\mathbb{R}$ with its standard smooth structure it suffices to construct a smooth function f on M with df nowhere 0. Such a function is then strictly increasing or strictly decreasing and thus (by the inverse function theorem) is a diffeomorphism of M onto its image $f(M)$ which is a (possibly infinite) open interval. Since any such interval is diffeomorphic to $\mathbb{R}$ with its standard smooth structure, this gives the desired conclusion.

In order to find such a function f it is sufficient to find a 1-form ω which is nowhere 0 since we may then integrate ω to get f with $df = \omega$. To find ω, observe that we may take the sets U in our atlas to be connected and we may take the coordinate maps to x_U to be strictly increasing by replacing x_U by $-x_U$ if necessary. We can then choose a locally finite covering $\mathcal{C}$ of M by coordinate charts, and write $\omega = \sum_{U\in\mathcal{C}} \zeta_U dx_U$ where ζ_U is a smooth partition of unity subordinate to the covering. Since for overlapping charts $U, V \in \mathcal{C}$ we have $dx_U/dx_V > 0$, it follows that ω is nowhere 0.

2) a) Define $F : (M_1 \times M_2) \times M_3 \to M_1 \times (M_2 \times M_3)$ by $F((P_1, P_2), P_3) = (P_1, (P_2, P_3))$. This map is clearly 1-1 and onto. The fact that it is smooth is just the obvious statement in local coordinates that the map $((x, y), z) \to (x, (y, z))$ from $(\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}) \times \mathbb{R}^{m_3}$ to $\mathbb{R}^{m_1} \times (\mathbb{R}^{m_2} \times \mathbb{R}^{m_3})$. Similarly F^{-1} is smooth. It is also easy to see that the map $G : M_1 \times M_2 \to M_2 \times M_1$ given by $G(P_1, P_2) = (P_2, P_1)$ is a smooth map and $G = G^{-1}$ so it is a diffeomorphism.

b) By definition an atlas for $M_1 \times M_2$ can be taken to be the product charts $U_1 \times U_2$ with the coordinate mapping $\phi = (\phi_1, \phi_2)$ where ϕ_1 defines coordinates in U_1 and ϕ_2 in U_2. Thus if we choose coordinates x near $P \in N$ and coordinates y near $\pi_1(f(P))$ and z near $\pi_2(f(P))$, then the local expression of f is given by $(y, z) = (y(x), z(x))$ and such a map is smooth if and only if $y(x)$ and $z(x)$ are smooth maps.

3) a) and b) Since $H(X, Y) = H(Y, X) + [X, Y](f)$ and $[X, Y]$ is a vector field we have $[X, Y](f) = 0$ at P. Thus $H(X, Y) = H(Y, X)$ at P. Since for any smooth function h defined near P, $Xh(P)$ depends only on the value of X at P, we see that XYf depends only on X at P and YXf depends only on Y at P. Thus $H(X, Y)$ depends only on X and Y at P.

c) Let N be the nullspace of H; that is,

$$N = \{v : \ H(v, w) = 0 \text{ for all } w \in T_P M\}.$$

Let $e_1, \dots e_r$ be a basis for N and let V be a complementing subspace. It then follows that H is nondegenerate on V and so the argument in Problem 6a below produces a basis $e_{r+1}, \dots, e_m$ so that $H(e_i, e_j) = \lambda_i \delta_{ij}$ where λ_i is 1 or -1 for $r+1 \leq i, j \leq m$. The basis $e_1, \dots, e_m$ then satisfies the desired condition with $\lambda_i = 0$ for $i = 1, \dots, r$.

If $v_1, \dots, v_m$ is another basis of this type, then those v_i for which $\lambda_i = 0$ lie in N and they must form a basis for N. Thus we may assume they form the set $v_1, \dots, v_r$. If we assume that there are p of the basis vectors $e_{r+1}, \dots, e_{r+p}$ for which $\lambda_i = 1$, then we claim that there are exactly p of the vectors $v_1, \dots, v_m$ with that property. If there were p' such vectors with $p' > p$ we could find a nonzero vector v which is a linear combination of those vectors which satisfies $H(v, w) = 0$ for all $w \in N$ and $H(v, v_i) = 0$ for $i = r+1, \dots, r+p$. It follows that v can be expressed as a linear combination of $e_{r+p+1}, \dots, e_m$ and therefore $H(v, v) < 0$. On the other hand v was chosen as a linear combination of $v_{r+1}, \dots, v_{r+p'}$ which implies $H(v, v) > 0$. This contradiction implies $p' \leq p$ and reversing the roles of the two bases implies the opposite inequality. Therefore $p' = p$ and hence the bases have the same number of 0s, 1s, and -1s.

4) a) In a neighborhood of P the distribution can be defined as the linear span of the coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^r$, and by translation of coordinates we may assume that $x(P) = 0$. It follows that the integral submanifold through P is the submanifold defined by $x^{r+1} = \ldots = x^m = 0$. In the local coordinates the horizontal curve $\gamma(t)$ is defined by the curve $x = x(t)$ and the fact that γ is horizontal is the condition that $dx^i/dt = 0$ for $i = r+1, \dots, m$. Since we may assume that $\gamma(0) = P$, we have $x^{r+1}(t) = \ldots = x^m(t) = 0$ for all t. Since this argument may be applied in a neighborhood of any point of γ, it follows that γ lies in the integral submanifold containing P as long as this submanifold continues to extend.

b) If (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points, we construct a piecewise smooth horizontal curve in three parts. To join (x_1, y_1, z_1) to (x_2, y_1, z_1) we may take the line parallel to the x-axis. We now observe that for a curve of the form $(x_2, ty_2 + (1-t)y_1, z(t))$ for $0 \leq t \leq 1$ to be horizontal with $z(0) = z_1$ we must have $z(t) = z_1 + x_2(y_2 - y_1)t$ since $dz = x dy$ along the curve. Thus we may join (x_2, y_1, z_1) to the point (x_2, y_2, z_1') where $z_1' = z_1 + x_2(y_2 - y_1)$. To join (x_2, y_2, z_1') to (x_2, y_2, z_2), we observe that any such horizontal curve would project to a closed curve in the xy-plane, and along the curve we would have $dz = xdy$. Since the integral of xdy around a closed curve in the plane represents the enclosed signed area (negative if the curve is clockwise around a region), we may choose a circle in the plane of signed area $z_2 - z_1'$ and construct a horizontal curve which projects onto the circle and begins at (x_2, y_2, z_1'). This curve will then end at the desired point (x_2, y_2, z_2).

5) a) Note that if X and Y are vector fields with $Y = F_*(X)$, then if $P \in M$ and φ is a smooth function defined near $F(P)$, we have $Y(\varphi) \circ F = X(\varphi \circ F)$ near P. Thus we have $(Y_1(Y_2(\varphi))) \circ F = X_1(Y_2(\varphi) \circ F) = X_1 X_2(\varphi \circ F)$, and similarly $Y_2 Y_1(\varphi) \circ F = X_2 X_1(\varphi \circ F)$. Subtracting we find $[Y_1, Y_2](\varphi) \circ F = [X_1, X_2](\varphi \circ F)$. This implies $F_*([X_1, X_2]) = [Y_1, Y_2]$.

b) Given any point $P \in M$, we may find a neighborhood U of P so that the restriction of F to U is an embedding onto a coordinate plane. It follows that there are smooth vector fields X_i defined in U so that $F_*(X_i) = Y_i$ for $i = 1, 2$. From part a we have $[Y_1, Y_2] = F_*([X_1, X_2])$, and therefore $[Y_1, Y_2]$ is tangent to $F(M)$.

With U as above we observe that the condition that $Y_i = Z_i$ on $F(M)$ implies that $F_*(X_i) = Y_i = Z_i$ and therefore $[Y_1, Y_2] = F_*([X_1, X_2]) = [Z_1, Z_2]$ and thus $[Y_1, Y_2] = [Z_1, Z_2]$ along $\Sigma = F(M)$.

6) a) The proof is by induction on m. For $m = 1$, any nonzero vector $v \in V$ must satisfy $g(v, v) \neq 0$, and therefore $e_1 = (|g(v, v)|)^{-1/2} v$ is the desired basis.

Assume the conclusion is true in dimension $m - 1$ and consider a vector space V of dimension m with a nondegenerate form g. Given any nonzero vector v, there is a vector w so that $g(v, w) \neq 0$ since g is nondegenerate. It follows that $g(v + w, v + w) - g(v, v) - g(w, w) \neq 0$, so there is a vector e with $g(e, e) \neq 0$. Let $e_1 = (|g(e, e)|)^{-1/2} e$ so that $g(e_1, e_1)$ is either 1 or -1. Now let $W = \{v \in V : \ g(v, e_1) = 0\}$. Clearly the dimension of W is $m - 1$, and we consider the restriction of g to W. We claim that this restriction is nondegenerate. To see this, suppose $v \in W$ with $g(v, w) = 0$ for all $w \in W$. Since $g(v, e_1) = 0$ and V is spanned by W together with e_1 it follows that $g(v, w) = 0$ for all $w \in V$ and hence $v = 0$ since g is nondegenerate on V. Therefore the restriction of g to W is nondegenerate and by the inductive assumption there is a basis $e_2, \ldots, e_m$ for W with $g(e_i, e_j) = \lambda_i \delta_{ij}$ for $2 \leq i, j \leq m$. Since this is also true for $i = 1$ and $1 \leq j \leq m$, the conclusion follows.

The proof that the number of plus and minus 1s is independent of basis is the same as that given in 3c.

b) By replacing g by $-g$ if necessary we may assume that $\nu \leq m-\nu$. Choose the basis so that $\lambda_i = 1$ for $i = 1, \ldots, \nu$, and observe that the subspace spanned by the vectors $e_1 + e_{\nu+1}, \ldots, e_\nu + e_{2\nu}$ is contained in the null cone. If there were a subspace of dimension larger than ν contained in the null cone, then it would contain a nonzero vector v with $g(v, e_i) = 0$ for $i = 1, \ldots, \nu$. Thus v can be written as $v = \sum_{i=\nu+1}^m a^i e_i$, and $g(v, v) = -\sum_{i=\nu+1}^m (a^i)^2 < 0$, a contradiction.

c) We have $I(v)(e_i) = \sum_{j=1}^m a^j \langle e_j, e_i \rangle$, so we have $I(v) = \sum b_i \omega^i$ where $b_i = \sum_{j=1}^m g_{ij} a^j$.

7) We first show that i and ii are equivalent. Clearly i implies ii, so suppose that ii holds. From the previous problem we may choose a basis $e_1, \ldots, e_m$ so that $g(e_i, e_j) = \lambda_i \delta_{ij}$ with $\lambda_i = 1$ for $1 \leq i \leq \nu$ and $\lambda_i = -1$ for $\nu + 1 \leq i \leq m$ where $1 < \nu < m$ (since g is neither positive nor negative definite). Let V_+ denote the linear span of $e_1, \ldots, e_\nu$ and V_- denote the span of $e_{\nu+1}, \ldots, m$. Given any vector $v \in V_+$ with $g(v, v) = 1$ and any vector $w \in V_-$ with $g(w, w) = -1$ we have $g(v + w, v + w) = 0$ and therefore $b(v + w, v + w) = 0$. It follows that $b(v, v) + 2b(v, w) + b(w, w) = 0$. Similarly $v - w$ is null for g and thus $b(v, v) - 2b(v, w) + b(w, w) = 0$. It follows that $b(v, w) = 0$ and $b(v, v) + b(w, w) = 0$. In particular, we see that for all $v \in V_+$ and $w \in V_-$ we have $b(v, w) = 0$. The restriction of b to V_+ may be diagonalized in an orthonormal basis, so by changing the basis $e_1, \ldots, e_\nu$ by an orthogonal transformation we may assume that $b(e_i, e_j) = \mu_i \delta_{ij}$ for $1 \leq i \leq \nu$. Similarly we may perform an orthogonal change of basis on V_- and assume that $b(e_i, e_j) = \mu_i \delta_{ij}$ for $1 \leq i \leq m$. From above we have $b(e_1, e_1) = -b(e_i, e_i)$ for $\nu + 1 \leq i \leq m$. Therefore we have $\lambda_i = -c$ for $\nu + 1 \leq i \leq m$. Similarly $\lambda_i = c$ for $1 \leq i \leq \nu$. It follows that $b = cg$.

Since i clearly also implies iii and iv, we need only show that iii implies ii and iv implies ii to finish the proof. Since the proofs are similar, we show that iii implies ii. To do this we let the orthonormal basis for g be chosen as above and define V_+ and V_- as above. Observe that if v is any vector with $g(v, v) = -1$ we may write $v = \sinh(t) v_+ + \cosh(t) v_-$ where $v_+ \in V_+$ with $g(v_+, v_+) = 1$ and $v_- \in V_-$ with $g(v_-, v_-) = -1$ and $t > 0$. Since $|b(v, v)| \leq C$ is bounded independent of t, we have

$$|b(\tanh(t) v_+ + v_-, \tanh(t) v_+ + v_-)| \leq C \cosh(t)^{-2}.$$

Letting t tend to infinity we conclude that $b(v_+ + v_-, v_+ + v_-) = 0$. Since *any* null vector is of this form it follows that $b(v, v) = 0$ for all null vectors v. (Note that we did not need the condition that b is nondegenerate in this problem!)