MMAT 5011 Analysis II
2016-17 Term 2
Assignment 5
Due date: Apr 18, 2017

You do not have to turn in the solution of optional problems. However, you are encouraged to try all the problems.

1. (Optional) Show that in an inner product space, $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for any $\alpha \in \mathbb{F}$.

2. Consider the subspace $Y = \{(z_1, z_2) \in \mathbb{C}^2 : 2z_1 + iz_2 = 0\} = \operatorname{span}\{(1, 2i)\} \subset \mathbb{C}^2$ and $x = (3+2i, 1+6i)$. Find the unique vector $z = (z_1, z_2) \in Y$ such that $x - z \in Y^\perp$. Hence, compute the distance between x and Y.

3. Show that $(Y^\perp)^\perp = \overline{Y}$ for a subspace Y of an Hilbert space H.

4. Define an inner product on $P_2(\mathbb{R})$ by
$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$
Apply the Gram-Schmidt process to $\{1, x, x^2\}$ in the given order to obtain an orthonormal basis of $P_2(\mathbb{R})$ under the inner product defined above.

5. The Legendre polynomial of order n, P_n, is a degree n polynomial defined by
$$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dt^n}[(t^2-1)^n].$$
   (a) By repeated application of integration by parts, show that for $m \geq n \geq 0$,
$$\int_{-1}^1 \left(\frac{d^n}{dt^n}[(t^2-1)^n]\right)\left(\frac{d^m}{dt^m}[(t^2-1)^m]\right) dt = \int_{-1}^1 \left(\frac{d^{n+m}}{dt^{n+m}}[(t^2-1)^n]\right)(1-t^2)^m dt$$
   (b) Let $I_m = \int_{-1}^1 (1-t^2)^m dt$. Show that for $m \geq 1$,
$$I_m = \frac{2m}{1+2m} I_{m-1}.$$
   (c) Show that $\left\{e_n(t) = \sqrt{\frac{2n+1}{2}} P_n(t) : n \geq 0\right\}$ is a total orthonormal set in $L^2[-1, 1]$.

6. Let M be a subset of a Hilbert space H. Show that $\overline{\operatorname{span} M} = H$ if and only if for any $x, y \in H$, $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in M$ implies that $x = y$.

7. Let
$$f_0(x) = \frac{1}{\sqrt{2\pi}}, \quad f_n(x) = \frac{1}{\sqrt{\pi}}\cos nx, \quad g_n(x) = \frac{1}{\sqrt{\pi}}\sin nx \quad \text{for } n \geq 1.$$
Show that $\{f_m, g_n : m \geq 0, n \geq 1\}$ is a total orthonormal set in $L^2[-\pi, \pi]$.

Remark. *Note that since these functions are real-valued, they also form a total orthonormal set in the real version of* $L^2[-\pi,\pi]$.

8. Since $\{e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx} : n \in \mathbb{Z}\}$ is a total orthonormal set of $L^2[-\pi,\pi]$, we have

$$\sum_{n=-\infty}^{\infty} |\langle f, e_n\rangle|^2 = \|f\|^2 \text{ for any } f \in L^2[-\pi,\pi].$$

By considering $f(x) = x^k, k = 1, 2$, show that

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   (b) (Optional) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

9. (Optional) Let $(x_1, y_1), (x_2, y_2), \ldots, (x_n y_n) \in \mathbb{R}^2$ with $x_1 < x_2 < \ldots < x_n$. The least square problem is about finding the 'best' straight line $y = mx + c$ to fit the given points in the sense that to minimize

$$\sum_{k=1}^{n} [y_k - (mx_k + c)]^2$$

among all $m, c \in \mathbb{R}$. One can find this best line using an orthogonal projection.

   (a) Define an inner product on $P_{n-1}(\mathbb{R})$ by

$$\langle f, g\rangle = \sum_{k=1}^{n} f(x_k)g(x_k).$$

   Apply the Gram-Schmidt orthogonalization process and normalization to $\{1, x\}$ to obtain an orthonormal basis for W under the inner product defined above.

   (b) There exists a unique polynomial $f \in P_{n-1}(\mathbb{R})$ such that $f(x_k) = y_k$ for $k = 1, 2, \ldots n$. Find the orthogonal projection of f onto $P_1(\mathbb{R})$.

   (c) Using the shortest distance property of orthogonal projection, conclude that the best fitting line $y = mx + c$ is given by

$$m = \frac{(\sum x_k)(\sum y_k) - n\sum x_k y_k}{(\sum x_k)^2 - n\sum x_k^2} \quad \text{and} \quad c = \frac{\sum y_k - m\sum x_k}{n}.$$

10. (Optional) Let $a_n \in \mathbb{C}, n \geq 0$, be a sequence and $\sigma_n = \frac{1}{n+1}\sum_{i=0}^{n} a_i$ be the arithmetic mean for the first $n + 1$ terms.

   (a) Show that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} \sigma_n = L$.

   (b) Show that the converse of part (a) is not true by finding a divergent sequence (a_n) such that the corresponding (σ_n) converges.

Remark. *Using the result of part (a) and Fejér's theorem, one can deduce that if the Fourier series* f_n *of a* 2π*-periodic continuous function* f *converges at a point* $x \in \mathbb{R}$*, then* $\lim_{n\to\infty} f_n(x) = f(x)$.