MMAT 5011 Analysis II
2016-17 Term 2
Assignment 2
Suggested Solution

1. (a) Since A is measure zero, for any $\varepsilon > 0$, we can find finite/countable open intervals $(a_i, b_i), i = 1, 2, ..., n$ (n can be ∞), such that $A \subset \bigcup_i (a_i, b_i)$ and $\sum_i |b_i - a_i| < \varepsilon$. Thus $B \subset A \subset \bigcup_i (a_i, b_i)$ with $\sum_i |b_i - a_i| < \varepsilon$. Hence B is measure zero.

   (b) Same as in (a), for any $\varepsilon > 0$, we have finite/countable open intervals $(a_{i,j}, b_{i,j}), j = 1, 2, ..., n_i$ (n_i can be ∞), such that $A_i \subset \bigcup_j (a_{i,j}, b_{i,j})$ and $\sum_j |b_{i,j} - a_{i,j}| < \varepsilon/2^i$. Thus $\bigcup_i A_i \subset \bigcup_i \bigcup_j (a_{i,j}, b_{i,j})$ with $\sum_i \sum_j |b_{i,j} - a_{i,j}| < \sum_i \varepsilon/2^i = \varepsilon$. Hence $\bigcup_i A_i$ is measure zero.

2. Denote by $\mathcal{S}$ the set $\{c_1x_1 + c_2x_2 + ... + c_nx_n : c_i > 0 \text{ for } 1 \leq i \leq n\}$. By lemma 2.4-1, there exists some positive constant c such that

$$|c_1x_1 + c_2x_2 + ... + c_nx_n| \geq c(|c_1| + |c_2| + ... + |c_n|)$$

holds for all $c_i \in \mathbb{R}, i = 1, 2, ..., n$. Let $x = c_1x_1 + c_2x_2 + ... + c_nx_n \in \mathcal{S}$.
Choose $r = c \cdot \min\limits_{1\leq i\leq n} c_i$. Then for any $y = b_1x_1 + b_2x_2 + ... + b_nx_n \in B(x, r)$, we have

$$c(|c_1 - b_1| + |c_2 - b_2| + ... + |c_n - b_n|) \leq |x - y| < r = c \cdot \min_{1\leq i\leq n} c_i.$$

Hence, for any i, $|c_i - b_i| < c_i$ and so $b_i > 0$. Hence $y \in \mathcal{S}$ and $B(x, r) \subset \mathcal{S}$. This proves $\mathcal{S}$ is an open subset of X.

3. (a) Let $\sum_{i=1}^{\infty} x_i$ be an absolute convergent series in X. Then $\sigma_k = \sum_{i=1}^{k} \|x_i\|$ is a convergent sequence. Consequently, σ_k is a Cauchy sequence. Thus for any $\varepsilon > 0$, there exists $N > 0$ such that

$$|\sigma_m - \sigma_n| \leq \varepsilon, \text{ whenever } m, n > N.$$

   Hence (we assume $m < n$ here)

$$\|s_n - s_m\| = \left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\| = |\sigma_n - \sigma_m| \leq \varepsilon.$$

   This implies s_k is a Cauchy sequence in X. Since X is a Banach space (i.e. a complete normed space), the sequence s_k is convergent.

   (b) Let X be the subspace of l^1 consisting of all sequences with finitely many non-zero terms. In other words,

$$X = \{\vec{a} = (a_1, a_2, \ldots) : \exists N > 0 \text{ such that } a_i = 0 \ \forall i \geq N\}$$

   Consider $x_i = \dfrac{1}{i^2} e_i \in X$. Then the sequence

$$s_k = \sum_{i=1}^{k} x_i = (1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{k^2}, 0, 0, \ldots)$$

is not convergent in X while

$$\sigma_k = \sum_{i=1}^{k} \|x_i\| = \sum_{i=1}^{k} \frac{1}{i^2}$$

is convergent (in $\mathbb{R}$).

4. (a) Consider $x_n = (1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots) \in Y$. The limit $\lim_{n\to\infty} x_n = (\frac{1}{n})_{n=1}^{\infty} \notin Y$. Thus Y is not a closed subspace of the Banach space l^∞. By theorem 1.4-7, Y is not complete.

   (b) Note that c_0 is a closed in l^∞. Indeed, $c_0 = \overline{Y}$. To prove this, suppose $x = (x_1, x_2, \ldots) \in c_0$. Define $y_n \in Y$ by

$$y_{n,i} = \begin{cases} x_i & \text{if } i \le n; \\ 0 & \text{if } i > n. \end{cases}$$

   We want to show that $\lim_{n\to\infty} y_n = x$. For any ε, since $\lim_{i\to\infty} x_i = 0$, there exists $N > 0$ such that for all $i > N$, $|x_i| < \varepsilon$. It implies that $\|x - y_n\| < \varepsilon$ for all $n > N$. Hence $\lim_{n\to\infty} y_n = x$. This shows that $c_0 \subset \overline{Y}$.

   Next, suppose that $y_n \in Y$ and $\lim_{n\to\infty} y_n = x \in l^\infty$. We want to show that $x \in c_0$. For any ε, since $\lim_{n\to\infty} y_n = x$, there exists n such that $\|x - y_n\| < \varepsilon$. Since $y_n \in Y$, there exists $N > 0$ such that $y_{n,i} = 0$ for all $i > N$. Hence,

$$|x_i| = |x_i - y_{n,i}| \le \|y_n - x\| < \varepsilon$$

   for all $i > N$. This proves $\lim_{i\to\infty} x_i = 0$ and so $x \in c_0$. It shows that $\overline{Y} \subset c_0$.

   Hence, $c_0 = \overline{Y}$ is closed in the Banach space l^∞. By theorem 1.4-7, c_0 is complete.

5. ($\Rightarrow$) T is bounded implies that there exists a real constant $c > 0$ such that $\|T(x)\| \le c\|x\|$ for all $x \in X$. Since A is bounded, we can also find a real constant $M > 0$ such that $\|x\| \le M$ for all $x \in A$. Thus $\|T(x)\| \le c\|x\| \le cM$ for any $x \in A$.

   ($\Leftarrow$) Consider $A = \{x \in X, \|x\| = 1\}$. Since A is bounded, $T(A)$ is also bounded. Hence, $\|T(x)\| \le N$ for some $N > 0$ and all $x \in A$. For any nonzero $x \in X$, $\frac{x}{\|x\|}$ is in A. Thus $\|T(\frac{x}{\|x\|})\| \le N \Leftrightarrow \|T(x)\| \le N\|x\|$, i.e., T is bounded.

6. Consider $X = l^1$ and $T : X \to X$ be defined by $T((x_i)_{i=1}^{\infty}) = \left(\left(1 - \frac{1}{i}\right) x_i\right)_{i=1}^{\infty}$.

   For any nonzero $x \in X$, $\left|\left(1 - \frac{1}{i}\right) x_i\right| < |x_i|$ for some i. Hence

$$\|T(x)\| = \textstyle\sum_i \left| \left(1 - \frac{1}{i}\right) x_i\right| < \sum_i |x_i| = \|x\|.$$

   In particular, $\|T\| \le 1$. To show $\|T\| = 1$, consider $x_n = e_n = (0, ..., 0, 1, 0, ...)$ where the only non-zero term is the n-th term . Then $\|x_n\| = 1, \|T(x_n)\| = 1 - \frac{1}{n} \to 1$ and so $\|T\| = \sup_{\|x\|=1} \|T(x)\| \ge 1$. Hence $\|T\| = 1$.

7. Let $p_n(x) = (n+1)x^n$. Then $p_n'(x) = n(n+1)x^{n-1}$.

$$\|p_n\| = \int_0^1 |(n+1)x^n| \, dx = 1.$$

$$\|p_n'\| = \int_0^1 |n(n+1)x^{n-1}|\,dx = n+1.$$

$$\Rightarrow \|T\| = \sup_{\|p\|=1} \|T(p)\| \geq \|T(p_n)\| = \|p_n'\| = n+1$$

for all n, thus T is unbounded.

8. Let $[0,1] \subset \bigcup (a_i, b_i)$. Since $[0,1]$ is compact, we may assume the open cover consists of finitely many intervals. Also, by renaming the indexes if necessary, we may assume that $0 \in (a_1, b_1), b_1 \in (a_2, b_2), b_3 \in (a_3, b_3), \ldots, b_{n-1} \in (a_n, b_n)$ and $1 \in (a_n, b_n)$. Hence,

$$\sum (b_i - a_i) \geq \sum_{i=1}^{n} (b_i - a_i) = b_n + \sum_{i=1}^{n-1} (b_i - a_{i+1}) - a_1 > 1 + 0 - 0 = 1.$$

It shows that $[0,1]$ is not measure zero.

9. (a) The sequence is not Cauchy in $\|\cdot\|_\infty$. For any integer $N > 0$, choose $m = N+1, n = 2N+2$. Note $m, n > N$ and

$$|f_n(\frac{1}{2N+2}) - f_m(\frac{1}{2N+2})| = |1 - \frac{N+1}{2N+2}| = \frac{1}{2}.$$

Thus $\|f_n - f_m\|_\infty \geq \frac{1}{2}$. Since N is arbitrary, (f_n) is not Cauchy (also not convergent).

(b) For any $\varepsilon > 0$, let $N > \frac{1}{\varepsilon}$, for all $n > m > N$,

$$\int_{-1}^{1} |f_n(x) - f_m(x)|\,dx = \int_0^{\frac{1}{m}} |f_n(x) - f_m(x)|\,dx \leq \int_0^{\frac{1}{m}} 1\,dx = \frac{1}{m} < \varepsilon$$

Thus (f_n) is Cauchy. However, f_n L^1-converges to

$$f(x) = \begin{cases} 0, & x \in [-1,0] \\ 1, & t \in (0,1] \end{cases},$$

which is not in $C[-1,1]$, (f_n) is not convergent in $C[-1,1]$.

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