**MMAT 5011 Analysis II**
**2016-17 Term 2**
**Assignment 1**
**Suggested Solution**

1. (a) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\det A = \det B = 0$, while $\det(A+B) = 1$.
Thus it is not a vector subspace.

(b) For any matrix $(a_{ij}), a_{ij} > 0$, $(-a_{ij})$ is not in the set. Thus it is not a vector subspace.

(c) Denote by $\mathcal{S}$ the subset of skew-symmetric matrices.
1) $0 \in \mathcal{S}$;
2) For any $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathcal{S}, \lambda \in \mathbb{R}$, $\lambda A = \begin{pmatrix} 0 & -\lambda a & -\lambda b \\ \lambda a & 0 & -\lambda c \\ \lambda b & -\lambda c & 0 \end{pmatrix} \in \mathcal{S}$;
3) For any $A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix}$,
$A + B = \begin{pmatrix} 0 & a+d & b+e \\ -(a+d) & 0 & c+f \\ -(b+e) & -(c+f) & 0 \end{pmatrix} \in \mathcal{S}$. Thus $\mathcal{S}$ is a vector subspace.
A basis for $\mathcal{S}$ is $\left\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \right\}$.

2. For $\|z\|_2$:
1) It is obvious that $\|z\|_2 \geq 0$.

2)
$$\|z\|_2 = 0 \Leftrightarrow \sum_{i=1}^{n} |z_i|^2 = 0 \Leftrightarrow |z_i| = 0, 1 \leq i \leq n \Leftrightarrow z_i = 0, 1 \leq i \leq n \Leftrightarrow z = 0$$

3) For any $\lambda \in \mathbb{C}$,
$$\|\lambda z\|_2 = \sqrt{\sum_{i=1}^{n} |\lambda z_i|^2} = \sqrt{\sum_{i=1}^{n} |\lambda|^2 |z_i|^2} = |\lambda| \sqrt{\sum_{i=1}^{n} |z_i|^2} = |\lambda| \|z\|_2$$

4) From the finite *Minkowski* inequality,
$$\sqrt{\sum_{i=1}^{n} |x_i + y_i|^2} \leq \sqrt{\sum_{i=1}^{n} |x_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i|^2}.$$

Thus for $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$ for any $x, y \in \mathbb{C}^n$.

From 1) 2) 3) 4), $\|z\|_2$ defines a norm on $\mathbb{C}^n$.

For $\|z\|_{1/2}$, take $n=2$, $x=(1,0), y=(0,1)$,
$\|x+y\|_{1/2} = \|(1,1)\|_{1/2} = 4 > \|x\|_{1/2}+\|y\|_{1/2} = 2$. It violates the triangle inequality, thus is not a norm.

3. (a) Let $\mathbf{x} = (x_1, x_2, ...) \in l^p$. Note that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ implies $\lim_{i\to\infty} |x_i|^p = 0$. For $\varepsilon = 1$, we can find $n > 0$ such that $|a_i| < 1$ for all $i > n$. Consequently, $|a_i|^q < |a_i|^p$ for all $i > n$. Thus

$$\sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{n} |x_i|^q + \sum_{i=n+1}^{\infty} |x_i|^q \leq \sum_{i=1}^{n} |x_i|^q + \sum_{i=n+1}^{\infty} |x_i|^p < \infty.$$

This shows $x \in l^q$ and hence $l^p \subset l^q$.

(b) Let $\mathbf{x} = (x_i)_{i=1}^{\infty}, x_i = \frac{1}{i^p}$. Then

$$\sum_{i=1}^{\infty} |x_i|^p = \sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

while

$$\sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{\infty} \frac{1}{i^{\frac{q}{p}}} < \infty.$$

This implies the inclusion $l^p \subset l^q$ is proper.

4. Let $x_n = \frac{a_n}{\sqrt{b_n}}, y_n = \sqrt{b_n}$, by Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n} x_n y_n\right)^2 \leq \left(\sum_{i=1}^{n} x_n^2\right)\left(\sum_{i=1}^{n} y_n^2\right)$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} \frac{a_n}{\sqrt{b_n}}\sqrt{b_n}\right)^2 \leq \left(\sum_{i=1}^{n} \left(\frac{a_n}{\sqrt{b_n}}\right)^2\right) \cdot \left(\sum_{i=1}^{n} \left(\sqrt{b_n}\right)^2\right)$$

$$\Leftrightarrow \frac{(a_1 + a_2 + ... + a_n)^2}{b_1 + b_2 + ... + b_n} \leq \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + ... + \frac{a_n^2}{b_n}$$

Let $n=3$ and $a_1 = x, a_2 = y, a_3 = z, b_1 = 3, b_2 = 4, b_3 = 5$, we get the required inequality.

5. Since $\mathbf{x}, \mathbf{y} \in l^{\infty}$, we have for any $i$

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|\mathbf{x}\|_{\infty} + \|\mathbf{y}\|_{\infty}.$$

Taking supremum, we have

$$\|\mathbf{x}+\mathbf{y}\|_{\infty} = \sup_i |x_i + y_i| \leq \|\mathbf{x}\|_{\infty} + \|\mathbf{y}\|_{\infty} < \infty.$$

Thus $\mathbf{x}+\mathbf{y} \in l^{\infty}$.

6. By triangle inequality,

$$\begin{aligned} \|\mathbf{x}\|_p = \|(\mathbf{x}-\mathbf{y})+\mathbf{y}\|_p \le \|\mathbf{x}-\mathbf{y}\|_p + \|\mathbf{y}\|_p \quad &\Rightarrow \quad \|\mathbf{x}\|_p - \|\mathbf{y}\|_p \le \|\mathbf{x}-\mathbf{y}\|_p. \\ \|\mathbf{y}\|_p = \|(\mathbf{y}-\mathbf{x})+\mathbf{x}\|_p \le \|\mathbf{y}-\mathbf{x}\|_p + \|\mathbf{x}\|_p \quad &\Rightarrow \quad \|\mathbf{x}\|_p - \|\mathbf{y}\|_p \ge -\|\mathbf{x}-\mathbf{y}\|_p. \end{aligned}$$

Combining the above two inequalities, we get

$$\left|\|\mathbf{x}\|_p - \|\mathbf{y}\|_p\right| \le \|\mathbf{x}-\mathbf{y}\|_p.$$

7. Since for any $0 < \alpha < 1$ and $x, y > 0$,

$$\log((1-\alpha)x + \alpha y) \ge (1-\alpha)\log x + \alpha \log y.$$

Substituting $x = a^p, y = b^q, \alpha = \frac{1}{q}$, we have

$$\log\left((1-\frac{1}{q})a^p + \frac{1}{q}b^q\right) \ge \left(1-\frac{1}{q}\right)\log a^p + \frac{1}{q}\log b^q$$

$$\Leftrightarrow \log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right) \ge \log ab.$$

Since the exponential funtion $e^x$ is increasing, we have

$$e^{\log\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)} \ge e^{\log ab},$$

i.e.,

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$

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