Tutorial 4

- More about parametric curves
- Limit

① (a) Let $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^2$ be a curve defined as follows:

$$\forall t \in \mathbb{R}, \quad \vec{\alpha}(t) = (t^3, t^2).$$

(i) Sketch the curve in $\mathbb{R}^2$.

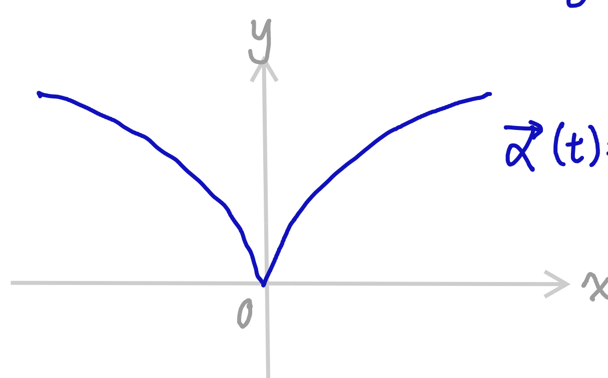
(ii) Are the component functions of $\vec{\alpha}$ smooth?

(iii) Does the curve in (i) look smooth?

(b) What is $\vec{\gamma}(t) := (1+t^2, 2-3t^2)$?

Ans: 1 (a) (i) $\forall t \in \mathbb{R}$, let $x := t^3$, $y := t^2$.

$$t = \sqrt[3]{x} \Rightarrow y = t^2 = x^{\frac{2}{3}}$$



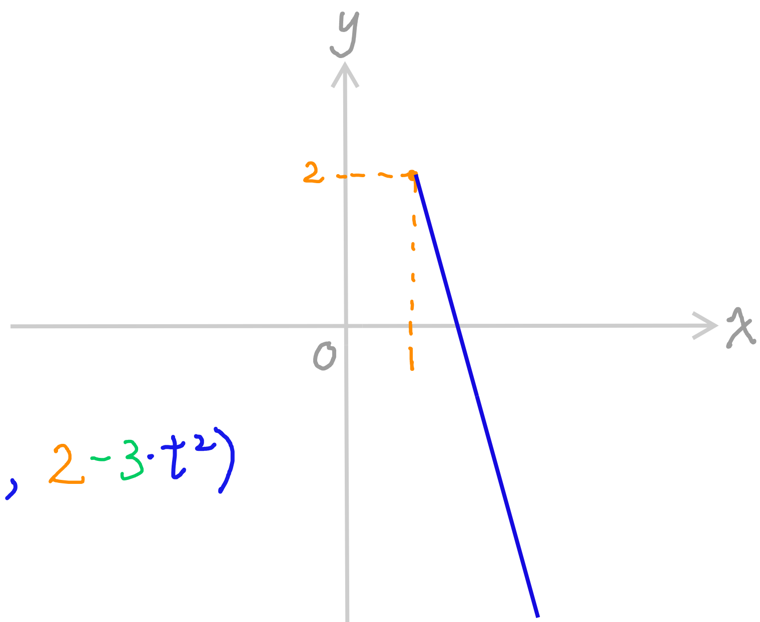
(ii) Yes. $\vec{\alpha}'(t) = (3t^2, 2t)$

(iii) No. There seems to be a cusp at $t=0$.

Remark: A "smooth parametric curve" may not be a "smooth curve" that we usually imagine. This peculiarity may be removed if we add the condition that $\vec{\alpha}'(t) \neq \vec{0}$ $\forall t \in \mathbb{R}$.

① (b) What is $\vec{\gamma}(t) = (1+t^2, 2-3t^2)$?

Ans: A half line.



$\vec{\gamma}(t) = (1+1\cdot t^2, 2-3\cdot t^2)$

② Use ε-δ definition of limit to prove Squeeze Theorem:

Let $\Omega \subseteq \mathbb{R}^n$ and let $\vec{a}$ be a cluster point of Ω.
If $f, g, h : \Omega \to \mathbb{R}$ are functions such that

$$(\forall \vec{x} \in \Omega, \ g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x}))$$

and $$\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) =: L,$$

then $$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$$

Ans: Let $\Omega \subseteq \mathbb{R}^n$ and let $\vec{a}$ be a cluster point of Ω.
Suppose $f, g, h : \Omega \to \mathbb{R}$ are functions such that

$$(\forall \vec{x} \in \Omega, \ g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x})) \quad \text{and}$$

$$\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) =: L$$

Want: $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$

Recall Defn (ε-δ): $f: \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$
We say that $\boxed{\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L}$ if
$\forall \varepsilon > 0, \ \exists \delta > 0$ such that
if $\vec{x} \in \Omega$ and $0 < \|\vec{x} - \vec{a}\| < \delta$,
then $|f(\vec{x}) - L| < \varepsilon$

(continue) Let $\varepsilon > 0$.

Since $L = \lim_{\vec{x} \to \vec{a}} g(\vec{x})$, $\exists\, \delta_1 > 0$ such that if $\vec{x} \in \Omega$ and $0 < \|\vec{x} - \vec{a}\| < \delta_1$, then $|g(\vec{x}) - L| < \varepsilon$.

Since $L = \lim_{\vec{x} \to \vec{a}} h(\vec{x})$, $\exists\, \delta_2 > 0$ such that if $\vec{x} \in \Omega$ and $0 < \|\vec{x} - \vec{a}\| < \delta_2$, then $|h(\vec{x}) - L| < \varepsilon$.

Now take $\delta := \min\{\delta_1, \delta_2\}$.

Suppose $\vec{x} \in \Omega$ and $0 < \|\vec{x} - \vec{a}\| < \delta$.

Since $g(\vec{x}) \le f(\vec{x}) \le h(\vec{x})$,

$$|f(\vec{x}) - L| \le \max\{|g(\vec{x}) - L|, |h(x) - L|\} < \varepsilon$$

$\therefore \lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$.

More detailed reason:

Case 1: Suppose $f(\vec{x}) - L \geq 0$.

$(f(\vec{x}) \le h(\vec{x})$ and $f(\vec{x}) - L \ge 0)$

$\Rightarrow 0 \le f(\vec{x}) - L \le h(\vec{x}) - L$

$\Rightarrow |f(\vec{x}) - L| \le |h(\vec{x}) - L|$

Case 2: Suppose $f(\vec{x}) - L < 0$.

$(g(\vec{x}) \le f(\vec{x})$ and $f(\vec{x}) - L < 0)$

$\Rightarrow g(\vec{x}) - L \le f(\vec{x}) - L < 0$

$\Rightarrow |f(\vec{x}) - L| \le |g(\vec{x}) - L|$

③ Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function defined as follows:

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < y < x^2 \text{ and } x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\lim_{(x,y) \to (0,0)} f(x,y) \neq 0$.

(b) Show that $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

Ans: 3(a) Take $\varepsilon_0 := \frac{1}{2}$. Let $\delta > 0$.

(want: $\exists (x_0, y_0) \in \mathbb{R}^2$ with $0 < \|(x_0,y_0)-(0,0)\| < \delta$ such that $|f(x_0,y_0) - 0| \geq \varepsilon_0 = \frac{1}{2}$)

By Density Theorem, $\exists\, q \in \mathbb{Q}$ s.t. $0 < q < \frac{1}{10}\min\{\delta, 1\}$.

Take $(x_0, y_0) := (q, \frac{1}{3}q^2) \in \mathbb{R}^2$. Note that

$$0 < \|(x_0,y_0)-(0,0)\| = \sqrt{q^2 + \frac{1}{9}q^4} = q\sqrt{1+\frac{1}{9}q^2} < \left(\frac{\delta}{10}\right)\sqrt{1+\frac{1}{9}} < \delta$$

Since $0 < y_0 < x_0^2$ and $x_0, y_0 \in \mathbb{Q}$,

$$|f(x_0,y_0) - 0| = |1 - 0| \geq \frac{1}{2} = \varepsilon_0$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) \neq 0$.

3(b) It suffices to show that

$$\forall L \in \mathbb{R}, \lim_{(x,y)\to(0,0)} f(x,y) \neq L$$

<u>Case 1:</u> $L = 0$. The result is shown in (a).

<u>Case 2:</u> $L \neq 0$. Take $\varepsilon_0 := \frac{|L|}{2} > 0$. Let $\delta > 0$.

Take $(x_0, y_0) := (0, \frac{\delta}{2}) \in \mathbb{R}^2$.

Note that $0 < \|(x_0,y_0)-(0,0)\| = \frac{\delta}{2} < \delta$.

$$|f(x_0,y_0) - L| = |0 - L| \geq \frac{|L|}{2} = \varepsilon_0$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y) \neq L$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist in $\mathbb{R}$.