Tutorial 10

$C^k \Rightarrow C^1$ $(k \geq 1)$

$\Downarrow$

(Total) differentiability

($\exists$ "good" linear approximation "locally")

$\Downarrow$

Chain rule

(higher order approximation)
Taylor's expansion of multivariable function

Implicit differentiation

$f^{-1}(c) \perp \nabla f(\vec{a})$

$D_{\vec{u}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$

(Taylor's Theorem: "better" approximation) — helps to prove → Second derivative test

Tools of $\boxed{\text{Finding Extrema}}$ ☆☆☆☆☆☆☆☆☆☆☆

- First derivative test
- Second derivative test
- Lagrange Multiplier(s)

① Recall First derivative test:

Let B be an arbitrary subset of $\mathbb{R}^n$. Let $b \in \mathrm{Int}(B)$.
(*) If a function $f: B \to \mathbb{R}$ attains a local extremum at b, then b is a critical point of f.

(a) State the definition of critical point under this context.

(b) Does the existence of critical point imply the existence of extrema ?

(c) Is it true that

Converse of (*): if $b \in \mathrm{Int}(B)$ is a critical point of f, then f attains a local extremum at b ?

(d) Why (and when) is 1st derivative test useful ?

Ans: 1 (a) A point $b \in \mathbb{R}^n$ is a critical point of $f: B \subseteq \mathbb{R}^n \to \mathbb{R}$ iff $b \in Int(B)$ and ($\nabla f(b)$ does not exist or $\nabla f(b) = \vec{0}$).

(b) No. A counter example may be

$$f: \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \ldots, x_n) \mapsto x_1^3, \text{ and take } b = (0, \ldots, 0).$$

$$\nabla f(x_1, \ldots, x_n) = (3x_1^2, 0, \ldots, 0)$$
$$\nabla f(b) = (0, 0, \ldots, 0) = \vec{0}$$

Thus, b is a critical point of f.

(Ex: Show that f does not have any local/global extrema.)

(c) No. See counter example in (b)

(Idea:) (d) The reason why 1st derivative test is useful may probably be understood via the contrapositive of (*):

"If $b \in Int(B)$ is <u>not</u> a critical point of f, then f <u>does not</u> attain a local extremum at b."

(Note: $(P \Rightarrow Q) \equiv ((\sim Q) \Rightarrow (\sim P))$
$\not\equiv$
$(Q \Rightarrow P)$)

1st derivative test helps to detect local extrema by <u>ruling out</u> the points in the interior of domain(f) where f <u>does not</u> attain local extremum.

Intuitively speaking, when domain(f) has "many" interior points that are not critical points, 1st derivative test should be useful.

② Recall Taylor's Thm for multivariable function:

Let Ω be an open subset of $\mathbb{R}^n$. Let $a \in \Omega$.
If a function $f : \Omega \to \mathbb{R}$ is C^k,
then
$$\lim_{x \to a} \frac{f(x) - P_{a,k}(x)}{\|x - a\|^k} = 0$$

where $P_{a,k}(x) := f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) \cdot (x_i - a_i) + \cdots$
$$+ \frac{1}{k!} \sum_{i_1, \dots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a) \cdot (x_{i_1} - a_{i_1}) \cdot \ldots \cdot (x_{i_k} - a_{i_k})$$
is called the k-th order Taylor polynomial of f at a.

(a) Compare the meanings of statements

△1 "$\forall k \in \mathbb{N}, \ \lim_{x \to a} \frac{(f(x) - P_{a,k}(x))}{\|x - a\|^k} = 0$"

and △2 "$\forall x \in B_r(a), \ \lim_{k \to \infty} (f(x) - P_{a,k}(x)) = 0$".

(b) How are statements △1 and △2 related?

Remarks:

△1 is the conclusion of Taylor's Thm if f is C^∞
(We say f is C^∞ or smooth iff all partial derivatives (of any order) exist & are cont.)

A function satisfying △2 for each $a \in \text{domain}(f)$ is called a real analytic function.

Ans: 2(a) Roughly speaking, $\triangle_1$ means that fixing any k, the k-th order Taylor polynomial $P_{a,k}$ is a "(very) good" approximation to f near the point a, while $\triangle_2$ means that fixing any x in some open ball (contained in Ω) containing $a \in \mathbb{R}^n$, where x may not be very close to a,

there exists Taylor polynomial centered at a (with high enough order) such that $P_{a,k}(x)$ is close to $f(x)$.

Rigorously: MATH 2060

2(b) Note that

$$\boxed{C^\infty \;\not\Longrightarrow\; \text{real analytic} \qquad C^\infty \Longleftarrow \text{real analytic}}$$

Therefore, roughly speaking,

$$\triangle_1 \not\Longrightarrow \triangle_2, \qquad \triangle_1 \Longleftarrow \triangle_2$$

An example that is C^∞ but not real analytic is

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \in \mathbb{R}\setminus\{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

(Ex. 6.4.10 (P.196) & Ex. 9.4.12 (P.287) in Bartle's "Introduction to Real Analysis")