Tutorial 1

This tutorial focuses mainly on the Cauchy-Schwarz inequality.

1 (a) Show the ($\Leftarrow$) part of

Cauchy-Schwarz inequality $\Leftrightarrow$ triangle inequality

(b) Is it true that

"Cauchy-Schwarz equality $\Leftrightarrow$ triangle equality" ?

Ans: (a) Suppose it is true that

$$\forall \vec{a}, \vec{b} \in \mathbb{R}^n, \ \|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|.$$

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$.

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

$$\|\vec{x}+\vec{y}\|^2 \leq (\|\vec{x}\| + \|\vec{y}\|)^2$$

$$(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\cdot\|\vec{y}\| + \|\vec{y}\|^2$$

$$\vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\cdot\|\vec{y}\| + \|\vec{y}\|^2$$

$$\vec{x}\cdot\vec{y} \leq \|\vec{x}\|\cdot\|\vec{y}\|$$

Since $\vec{x}, (-\vec{y}) \in \mathbb{R}^n$, we have

$$\|\vec{x} + (-\vec{y})\| \leq \|\vec{x}\| + \|\vec{y}\|$$

$$\|\vec{x} - \vec{y}\|^2 \leq (\|\vec{x}\| + \|\vec{y}\|)^2$$

$$\|\vec{x}\|^2 - 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \leq \|\vec{x}\|^2 + 2\|\vec{x}\|\cdot\|\vec{y}\| + \|\vec{y}\|^2$$

$$-\vec{x}\cdot\vec{y} \leq \|\vec{x}\|\cdot\|\vec{y}\|$$

As $|\vec{x}\cdot\vec{y}| := \begin{cases} \vec{x}\cdot\vec{y} & \text{if } \vec{x}\cdot\vec{y} \geq 0 \\ -\vec{x}\cdot\vec{y} & \text{if } \vec{x}\cdot\vec{y} < 0, \end{cases}$

we have $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\cdot\|\vec{y}\|$.

Ans: 1(b) No. Let $\vec{a}, \vec{b} \in \mathbb{R}^n$.

Claim: $\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\| \Rightarrow |\vec{a}\cdot\vec{b}| = \|\vec{a}\| \cdot \|\vec{b}\|$ and the converse is not true.

Proof of $(\Rightarrow)$: $\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$

$$\Leftrightarrow \exists \text{ non-negative } k \in \mathbb{R} \text{ s.t. } \vec{a} = k\vec{b}$$
$$\Rightarrow \exists\, k \in \mathbb{R} \text{ s.t. } \vec{a} = k\vec{b}$$
$$\Leftrightarrow |\vec{a}\cdot\vec{b}| = \|\vec{a}\| \cdot \|\vec{b}\|$$

Proof of $(\not\Leftarrow)$: Take $\vec{a} \in \mathbb{R}^n \setminus \{\vec{0}\}$ with $\vec{b} := -\vec{a}$.

$$|\vec{a}\cdot\vec{b}| = |\vec{a}\cdot(-\vec{a})| = \|\vec{a}\|^2$$

However, $\|\vec{a}+\vec{b}\| = \vec{0}$
$$\neq 2\|\vec{a}\| = \|\vec{a}\| + \|\vec{b}\|.$$

② Let $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Show that

$$\frac{(x_1+x_2+\ldots+x_n)^2}{n} \leq x_1^2 + x_2^2 + \ldots + x_n^2 .$$

Ans: Take $\vec{x} := (x_1, \ldots, x_n)$, $\mathbb{1} := (1, 1, \ldots, 1) \in \mathbb{R}^n$.

By Cauchy-Schwarz inequality,
$$|\vec{x}\cdot\mathbb{1}| \leq \|\vec{x}\| \cdot \|\mathbb{1}\|$$

$$|x_1+x_2+\ldots+x_n| \leq \sqrt{x_1^2+\ldots+x_n^2} \cdot \sqrt{1^2+\ldots+1^2}$$
$$= \sqrt{n} \cdot \sqrt{x_1^2+\ldots+x_n^2}$$

$$\therefore \frac{(x_1+x_2+\ldots+x_n)^2}{n} \leq x_1^2 + \ldots + x_n^2$$

③ Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
Show that there is a number $M \in \mathbb{R}$ such that for any $\hat{h} \in \mathbb{R}^m$, $\|T(\hat{h})\| \leq M \|\hat{h}\|$.

Ans: Let $[T]$ be the matrix representation of the linear transformation T with respect to the standard basis.

$$[T] = \begin{pmatrix} | & & | \\ T(e_1) & \cdots & T(e_m) \\ | & & | \end{pmatrix} = \begin{pmatrix} \text{---} & \vec{r}_1 & \text{---} \\ & \vdots & \\ \text{---} & \vec{r}_n & \text{---} \end{pmatrix} = (t_{ij})$$

where $\vec{r}_i$'s are the row vectors of $[T]$, and t_{ij} is the $(i,j)^{th}$ entry of $[T]$.

Take $M := \sqrt{\sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} t_{ij}^2}$. For each $\hat{h} \in \mathbb{R}^m$,

$$\|T(\hat{h})\| = \|[T]\hat{h}\| = \left\| \begin{pmatrix} \vec{r}_1 \cdot \hat{h} \\ \vdots \\ \vec{r}_n \cdot \hat{h} \end{pmatrix} \right\|$$

$$= \sqrt{|\vec{r}_1 \cdot \hat{h}|^2 + |\vec{r}_2 \cdot \hat{h}|^2 + \dots + |\vec{r}_n \cdot \hat{h}|^2}$$

(Cauchy-Schwarz inequality)

$$\leq \sqrt{\|\vec{r}_1\|^2 \|\hat{h}\|^2 + \|\vec{r}_2\|^2 \|\hat{h}\|^2 + \dots + \|\vec{r}_n\|^2 \|\hat{h}\|^2}$$

$$= \sqrt{\|\vec{r}_1\|^2 + \|\vec{r}_2\|^2 + \dots + \|\vec{r}_n\|^2} \cdot \|\hat{h}\| = M \|\hat{h}\|$$

Remark: The inequality in ③ could be used to show the continuity of linear maps $T: \mathbb{R}^m \to \mathbb{R}^n$ and differentiable maps $f: \mathbb{R}^m \to \mathbb{R}^n$.