THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 Stochastic Processes, 2020-21 Term 2

Take-home Midterm Test

Time and Date: 10:00am March 19 to 10:00am March 20

Answer all questions in both Part I and Part II (Total points: 120). Give adequate explanation and justification for all your computations and observations, and write your proofs in a clear and rigorous way.

Part I (100 points). Computations.

1. *(15 points)* Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $S = \{a, b, c\}$, transition matrix

$$P = \begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} \end{bmatrix} \end{array},$$

and the initial distribution $\pi = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$. Compute the following

(a) $P_a(X_1 = b, X_2 = b, X_3 = b, X_4 = a, X_5 = c)$,

(b) $P_c(X_1 = a, X_2 = c, X_3 = c, X_4 = a, X_5 = b)$,

(c) $P_a(X_1 = b, X_3 = a, X_4 = c, X_6 = b)$,

(d) $P(X_1 = b, X_2 = b, X_3 = a)$,

(e) $P(X_2 = b, X_5 = b, X_6 = b)$.

2. *(15 points)* Let $\{X_n\}_{n\geq 0}$ be a Markov chain with state space $S = \{x, y, z, w\}$ and transition matrix

$$P = \begin{array}{c} \begin{matrix} x & y & z & w \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{bmatrix} \end{array}.$$

(a) Compute $P(X_5 = z, X_6 = x, X_7 = z, X_8 = z | X_4 = y)$.

(b) Compute $E(f(X_5)f(X_6)|X_4 = w)$ for the function f with values $2, 3, 7$ and 3 at x, y, z and w respectively.

(c) For each $i, j \in S$, find ρ_{ij}, the probability that starting at i the chain ever visits j in finite time.

3. *(10 points)* Consider a Markov chain with state space $S = \{1, 2, 3\}$ and the transition matrix

$$
\begin{array}{c} \\ P = \end{array}
\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \end{array} \\
\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{3}{5} & \frac{1}{15} \end{bmatrix}
\end{array}.
$$

(a) For each $i = 1, 2, 3$ and all $k = 1, 2, \cdots$, compute the probabilities that starting at i, the first visit to 3 occurs at time k.

(b) For each $i = 1, 2, 3$, find the probability that starting at i, the chain never visits 3 at any positive time.

4. *(10 points)* Consider the Markov chain with state space $S = \{1, 2, \cdots, 10\}$ and transition matrix

$$
P = \begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{bmatrix}
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3}
\end{bmatrix}
\end{array}.
$$

(a) Draw the transition graph.

(b) State the decomposition of state space by finding all the irreducible closed sets of recurrent states as well as the set of transient states.

(c) Write down the canonical form of transition matrix by reordering states.

5. *(15 points)* Given a branching process with the offspring distribution

$$p_0 = 0.5, \; p_1 = 0.1, \; p_3 = 0.4.$$

(a) Determine the extinction probability ρ.

(b) Let $X_0 = 1$. What is the probability that the population is extinct in the second generation ($X_2 = 0$), given that it did not die out in the first generation ($X_1 > 0$)?

(c) Still let $X_0 = 1$. What is the probability that the population is extinct in the third generation, given that it was not extinct in the second generation?

6. *(15 points)* Let X_n, $n \geq 0$, denote the capital of a gambler at the end of the nth play. His strategy is as follows. If his capital is 4 dollars or more, then he bets 2 dollars which earn him $4, 3$ or 0 dollars with respective probabilities 0.25, 0.30 and 0.45. If his capital is $1, 2$ or 3 dollars, then he plays more conservatively, bets 1 dollar, and this earns him either 2 or 0 dollars with respective probabilities 0.45 and 0.55. When his capital becomes 0, he stops.

   (a) Let Y_{n+1} be the net earnings at the $(n+1)$th play, that is,
   $$X_{n+1} = X_n + Y_{n+1}.$$
   Compute
   $$P(Y_{n+1} = k | X_n = i), \quad i = 0, 1, \cdots ; k = -2, -1, 0, 1, \cdots .$$

   (b) Explain that $\{X_n\}_{n \geq 0}$ is a Markov chain.

   (c) Compute the transition probabilities for the chain.

   (d) Classify the states, either recurrent or transient.

7. *(20 points)* Let $\{X_n\}_{n \geq 0}$ be a Markov chain over $S = \{1, 2, \cdots, 7\}$ with the following transition matrix

$$P = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0.1 & 0.2 & 0.3 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.3 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0.6 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

   Determine the limit $\lim_{n \to \infty} P^n(x, y)$ for any $x, y \in S$.

Part II (20 points) Theories and Applications.

8. *(10 points)* Let $\{X_n\}_{n \geq 0}$ be a stochastic process taking values in a countable state space S. Suppose there exists an integer $K \geq 1$ such that
$$P(X_n = i_n | X_0 = i_0, \cdots, X_{n-1} = i_{n-1}) = P(X_n = i_n | X_{n-K} = i_{n-K}, \cdots, X_{n-1} = i_{n-1})$$
   for all $i_\ell \in S$ with $0 \leq \ell \leq n$ and for all $n \geq K$. In other words, given all the past, the future depends only on the last K values. Such a process is called a K-dependent chain. For $K = 1$, we have the ordinary Markov chains. Their theory can, however, be reduced to that of the ordinary Markov chains by the following procedure.

   For each $n \geq 0$, let
   $$Y_n = (X_n, X_{n+1}, \cdots, X_{n+K-1}).$$
   Then $\{Y_n\}_{n \geq 0}$ is a stochastic process taking values in the countable set $F = S^K = S \times \cdots \times S$. **Explain** that $\{Y_n\}_{n \geq 0}$ is an ordinary Markov chain.

9. *(10 points)* Let $\{X_n\}_{n\geq 0}$ be an irreducible Markov chain on the state space $S = \{1, \cdots, N\}$.

(a) Show that there exist $0 < C < \infty$ and $0 < \rho < 1$ such that for any states i, j,

$$P(X_m \neq j, m = 0, \cdots, n | X_0 = i) \leq C\rho^n, \quad \forall n.$$

(*Hint:* There exists a $\delta > 0$ such that for all i, the probability of reaching j some time in the first N steps, starting at i, is greater than δ. Why?)

(b) Show that (a) further implies $E(T_j) < \infty$, where T_j is the hitting time of j.

—THE END—