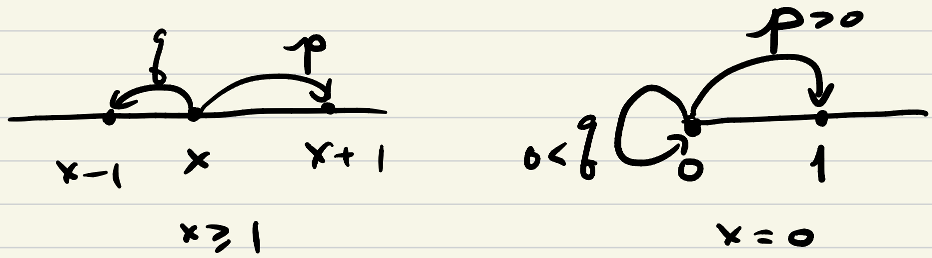
March 3:

Compute SD in case S is infinite.

example 1. Birth-death chain:

Recall:

$$S = \{0, 1, \cdots \}$$



$$p \cdot q > 0$$

$$p + q = 1$$

Question: determine the $\exists$ of SD.

$$\underset{\substack{\uparrow \\ \text{SD} \\ \text{to find}}}{\pi} = [\underbrace{\pi(0)}_{=x_0}, \underbrace{\pi(1)}_{=x_1}, \cdots]$$

$$= \underbrace{[x_0, x_1, \cdots]}_{\text{prob. row vector}}$$

s.t. $\pi P = \pi$

$$x_k = \sum_{i=0}^{\infty} x_i \cdot P(i,k)$$

k^{th}

$k = 0, 1, \cdots$

$$
[x_0, x_1, \cdots]\;
\begin{array}{c|c}
 & \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\
\hline
\begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} &
\begin{bmatrix}
q & p & & \\
q & 0 & \underset{(k-1,k)}{p} & \\
 & q & \underset{(k,k)}{0} & p \\
 & & \underset{(k+1,k)}{q} & \ddots
\end{bmatrix}
\end{array}
= [x_0, x_1, \cdots]
$$

Look at k^{th} component:

$k=0$: $\underset{\substack{\| \\ p+\cancel{q}}}{1}\cdot x_0 = \cancel{q x_0} + q x_1 \Rightarrow p x_0 = q x_1 \Rightarrow x_1 = \frac{p}{q} x_0$

$k \geq 1$: $\underset{\substack{\| \\ p+q}}{1}\, x_k = [x_{k-1}, x_k, x_{k+1}] \begin{bmatrix} p \\ 0 \\ q \end{bmatrix} = p x_{k-1} + q x_{k+1}$

$$\Rightarrow q x_{k+1} - p x_k \overset{k \to k-1}{=} q x_k - p x_{k-1}$$
$$= \cdots$$
$$= q x_1 - p x_0$$
$$= 0$$

$$\therefore\; x_k = \left(\frac{p}{q}\right)^1 x_{k-1} = \cdots = \left(\frac{p}{q}\right)^k x_0,$$

$1 + (k-1) = k$ $\qquad$ $k + 0 = k$

for $k = 0, 1, 2, \cdots$

$(x_0 = x_0)$

We have to require:

$$1 = \sum_{k=0}^{\infty} \underset{\substack{(\geq 0) \\ \text{if} \\ x_0 \geq 0}}{x_k} = \left[\sum_{k=0}^{\infty} \left(\frac{p}{q}\right)^k\right] x_0$$

Conclusion:

If $0 < \frac{p}{q} < 1$, then $\sum_{k\geq 0} = \frac{1}{1-\frac{p}{q}}$

$(\Leftrightarrow p < q)$

$\therefore \quad x_0 = 1 - \frac{p}{q} > 0$

In this case

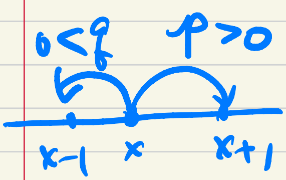
SD exists uniquely, given by

$$\pi = [x_0, x_1, \cdots]$$
$$= [x_0, \left(\tfrac{p}{q}\right)^1 x_0, \cdots]$$
$$= x_0 \left[1, \left(\tfrac{p}{q}\right), \left(\tfrac{p}{q}\right)^2, \cdots \right]$$
$$= \underbrace{\left(1 - \frac{p}{q}\right)}_{\in(0,\infty)} \left[1, \frac{p}{q}, \left(\frac{p}{q}\right)^2, \cdots \right]$$

If $\frac{p}{q} \geq 1$ (i.e. $p \geq q$):

NO SD!

Sum



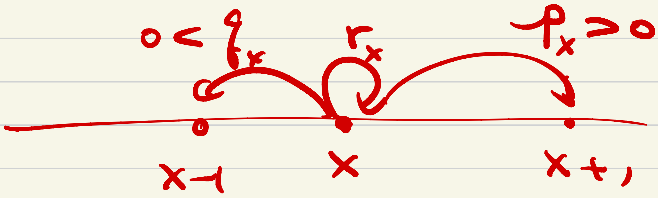
$p+q=1$

$p \geq q$	NO SD
$p < q$	$\exists!$ SD

Recall :

This chain recurrent iff $q \ge p$

Exercise : Repeat the argument to treat the $\exists$ of SD for a BD chain with a general transition function



$$p_x + r_x + q_x = 1$$

---

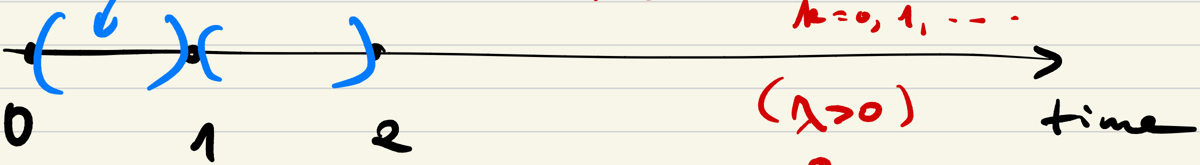
example 2 : Queuing chain

Setup

ξ arrivals : Poisson distribution

$$P(\xi = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

$k = 0, 1, \ldots$



$(\lambda > 0)$

arrival rate

Rule :

Each person (on the line waiting for the service at the beginning of a unit time) has prob. $q > 0$ to be served by the end of this unit time.

$$\{X_n\}_{n=0}^{\infty}$$

no of persons on the line at time $n=0, 1, 2, \dots$

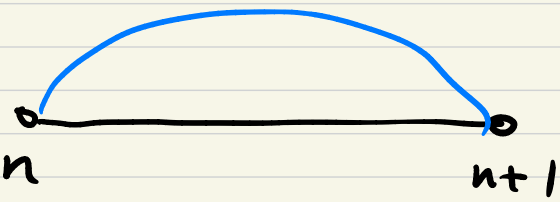
Question: Find transition prob. & SD?

March 8th:

$$P(x,y) = P(X_{n+1}=y \mid X_n = x)$$

$$x, y \in S = \{0, 1, 2, \dots\}$$

ξ arrivals (Poisson distribution with parameter $\lambda > 0$)



$X_n = x \geq 0$ $\qquad\qquad$ $X_{n+1} = y \geq 0$

each person from x at time n has prob. $\rho > 0$ to be served at time $n+1$

Observe:

$$X_{n+1} = \xi + Y_{n+1}$$

ξ: no. of arrivals in $(n, n+1)$

Y_{n+1}: no of persons from x (that are on the line at time n but still remains on the line at time $n+1$)

then

$$P(x,y) = P(\xi + Y_{n+1} = y \mid X_n = x)$$

$$= \sum_{z=0}^{\min\{x,y\}} P(\underbrace{\xi + Y_{n+1} = y}_{\Leftrightarrow\ \xi = y-z},\ \underline{Y_{n+1} = z} \mid X_n = x)$$

$0 \le z \le x$

$0 \le z \le y$

$$= \sum_{z=0}^{\min\{x,y\}} \underbrace{P(\xi = y-z)}_{= e^{-\lambda}\frac{\lambda^{y-z}}{(y-z)!}} \underbrace{P(Y_{n+1} = z \mid X_n = x)}_{= \binom{x}{z} [\text{success}]^z [\text{un-sucess}]^{x-z}}$$

$[\text{success}] \stackrel{\text{def.}}{=} p = 1-q$, $[\text{un-sucess}] = q$

$$= \binom{x}{z} p^z (1-p)^{x-z}$$

$$= \sum_{z=0}^{\min\{x,y\}} e^{-\lambda} \frac{\lambda^{y-z}}{(y-z)!} \cdot \binom{x}{z} p^z (1-p)^{x-z} \qquad \#$$

—— transition function $P(x,y)$.

How to find SD for this model :

IDEA :

Lemma 1 (proved later): If X_n is Poisson with rate $t > 0$, then Y_{n+1} is Poisson with rate pt. #

Lemma 2 (Exercise): X: Poisson, rate $= t_1 > 0$; Y: Poisson, rate $= t_2 > 0$ } indep't

Then

$$Z \overset{\text{def.}}{=} X + Y \text{ is Poisson with rate } = t_1 + t_2.$$

If so, we expect SD to be a Poisson (rate ?)

* Assume: X_0 is Poisson, rate $= t$ (?)

* Consider $X_1 = \underset{\substack{\text{Poisson} \\ \text{rate}=\lambda}}{\xi} + \underset{\substack{\text{(due to Lemma 1)} \\ \text{Poisson} \\ \text{rate} = pt}}{Y_1}$

$$\xRightarrow{\text{Lemma 2}} X_1 : \text{Poisson, rate} = \lambda + pt \boxed{= t}$$

Require:

$$\lambda + pt = t \Rightarrow \boxed{t} = \frac{\lambda}{1-p} = \frac{\lambda}{q}$$

* Consider $X_2 = \underset{\substack{\text{Poisson} \\ \text{rate}=\lambda}}{\xi} + \underset{\substack{\text{Poisson} \\ \text{rate}=pt}}{Y_2}$ $\Bigg] \Rightarrow X_2$: Poisson, rate $= \lambda + pt = t$.

* ....

Claim: If $t = \frac{\lambda}{q}$, then

$X_1, X_2, \cdots,$ are Poisson with the parameter t that is the same as the Poisson r.v. X_0.

$\therefore$ The Poisson distribution

$$\pi(x) = e^{-t}\frac{t^x}{x!} = e^{-\frac{\lambda}{\delta}}\frac{(\lambda/\delta)^x}{x!}, \quad x = 0, 1, \cdots$$

is the SD of the MC defined before.

Now, it remains to give

Proof of Lemma 1:

$$P(Y_{n+1} = y) \overset{y \geq 0}{=\!=} \sum_{x=y}^{\infty} P(Y_{n+1} = y, X_n = x)$$

$$= \sum_{x=y}^{\infty} \underbrace{P(Y_{n+1} = y \mid X_n = x)}_{=\binom{x}{y} p^y \underbrace{(1-p)}_{\text{"prob. served"}}{}^{x-y}} \underbrace{P(X_n = x)}_{= e^{-t}\frac{t^x}{x!}}$$

$$\binom{x}{y} = \frac{x!}{y!(x-y)!}$$

$$= \cdots = \frac{(pt)^y \cancel{e^{-t}}}{y!} \sum_{x=y}^{\infty} \underbrace{\frac{[t(1-p)]^{x-y}}{(x-y)!}}_{x-y=k}$$

$$= \sum_{k=0}^{\infty} \frac{[t(1-p)]^k}{k!}$$

$$= e^{\cancel{t}(1-p)}$$

$$= e^{-pt} \frac{(pt)^y}{y!}$$

Poisson, rate $= pt$. #