§3. More typical example

goal: 1° Birth-death chain
2°. Branching chain
3°. Queuing chain

Type # 1. Birth-death chain

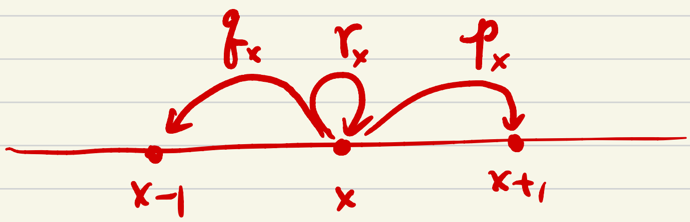
Setup:

MC: $\{X_t : t = 0, 1, 2, \cdots\}$

$S = \{0, \dots, d\} \quad (d \leq \infty)$

$$P(x,y) = \begin{cases} p_x \in [0,1] & \text{if } y = x+1 \\ r_x \in [0,1] & \text{if } y = x \\ q_x \in [0,1] & \text{if } y = x-1 \\ 0 & \text{otherwise} \end{cases}$$

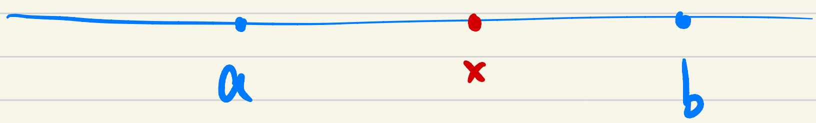
$$p_x + r_x + q_x = 1$$



Birth-death

Convention: $q_0 = 0$
(at bdy states)
$\{0, 1, \cdots, d\} \to 0, d \qquad p_d = 0$

a general question: $a, b \in S$ with $a < b$.



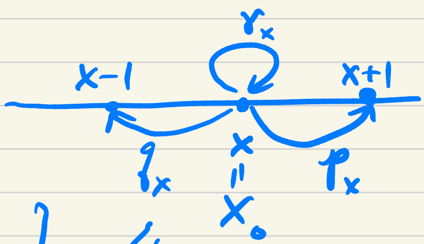
$$u(x) \overset{\text{def.}}{=} P_{x}\big(\underbrace{T_a < T_b}_{\text{the chain from } x \text{ visit } a \text{ earlier than } b}\big)$$

$$v(x) \overset{\text{def.}}{=} P_x\big(\underbrace{T_a > T_b}_{\dots}\big) = 1 - u(x)$$

Goal: Compute $u(x)$.

In fact

$p_x + r_x + q_x = 1$



$$u(x) = P_x(T_a < T_b)$$

$$= P_x(\overbrace{T_a<T_b}^{A}, \overbrace{X_1 = x+1}^{B}) + P_x(T_a<T_b, X_1 = x) + P_x(T_a<T_b, X_1 = x-1)$$

Only three cases

$P(A \cap B) = P(A|B)\,P(B)$

$$= \underbrace{P_x(T_a<T_b \mid X_1 = x+1)}_{= P_{x+1}(T_a<T_b) = u(x+1)} \underbrace{P_x(X_1 = x+1)}_{= p_x} + \underbrace{P_x(T_a<T_b \mid X_1 = x)}_{= u(x)} \underbrace{P_x(X_1 = x)}_{r_x} + \underbrace{P_x(T_a<T_b \mid X_1 = x-1)}_{= u(x-1)} \underbrace{P_x(X_1 = x-1)}_{q_x}$$

$$\therefore \boxed{u(x) = p_x\, u(x+1) + \underbrace{r_x}_{= 1 - p_x - q_x} u(x) + q_x\, u(x-1)}$$

$$\therefore (p_x + q_x)\, u(x) = p_x\, u(x+1) + q_x\, u(x-1)$$

$$\therefore \; p_x \left[u(x+1) - u(x)\right] = q_x \left[u(x) - u(x-1)\right]$$

$p_x > 0$

$$\therefore \; u(x+1) - u(x) = \frac{q_x}{p_x}\left[u(x) - u(x-1)\right] \quad \forall x$$

replace x by $a+1$

$$= \underbrace{\left(\frac{q_x}{p_x}\right)\left(\frac{q_{x-1}}{p_{x-1}}\right)\cdots\left(\frac{q_{a+1}}{p_{a+1}}\right)}_{=\frac{\gamma_x}{\gamma_a}} \left[u(a+1) - u(a)\right]$$

$u(a) \overset{?}{=}$

$$\frac{\gamma_x}{\gamma_a} = \frac{\text{“}\cancel{1 \cdots a} \cdot (a+1) \cdots x\text{”}}{\text{“}\cancel{1 \cdot \cdots \cdot a}\text{”}}$$

$a \quad x \quad b$

Def. :

$$\gamma_x \overset{\text{def.}}{=} \begin{cases} \left(\frac{q_1}{p_1}\right)\left(\frac{q_2}{p_2}\right)\cdots\left(\frac{q_x}{p_x}\right) & x = 1, 2, \ldots \\ 1 & x = 0 \text{ (convention)} \end{cases}$$

$$\sum_{x=a}^{b-1}(\ldots) \Rightarrow u(b) - u(a) = \frac{\sum_{x=a}^{b-1}\gamma_x}{\gamma_a}\left[u(a+1) - u(a)\right]$$

$$+\begin{cases} u(b) - \cancel{u(b-1)} \\ \cancel{u(b-1)} - \cancel{u(b-2)} \\ \vdots \\ \cancel{u(a+1)} - u(a) \end{cases}$$

Plug it back to replace $\underline{\underline{u(a+1) - u(a)}}$?

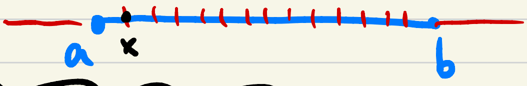
$$\therefore \; u(x+1) - u(x) = \frac{\gamma_x}{\cancel{\gamma_a}} \cdot \frac{\cancel{\gamma_a}}{\sum_{x=a}^{b-1}\gamma_x}\left[u(b) - u(a)\right] \quad \forall x$$

$$\underbrace{u(b) \overset{?}{=} 0 \quad u(a) \overset{?}{=} 1}_{= -1}$$

Fix $a < b$. we can let

$$\lim_{x\to a+} P_x(T_a<T_b) = 1 \Rightarrow u(a)=1$$

$$\lim_{x\to b-} P_x(T_a<T_b) = 0 \Rightarrow u(b)=0$$



think about the situation the unit of time is smaller and smaller.

$u(b)=0$

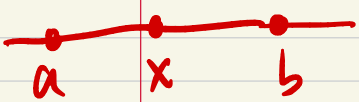
$$\underbrace{u(x+1)}-\underbrace{u(x)}_{?} = -\frac{\gamma_x}{\sum_{y=a}^{b-1}\gamma_y}, \quad x=a,\dots,b-1$$

$$\left.\begin{array}{l} x\to y \\ \sum_{y=x}^{b-1}(\dots) \end{array}\right\} \Rightarrow \underbrace{u(b)}_{=0}-u(x) = -\frac{\sum_{y=x}^{b-1}\gamma_y}{\sum_{y=a}^{b-1}\gamma_y}$$

$$\therefore \quad \boxed{u(x) = \frac{\sum_{y=x}^{b-1}\gamma_y}{\sum_{y=a}^{b-1}\gamma_y}, \quad x=a,\dots,b-1,b}$$

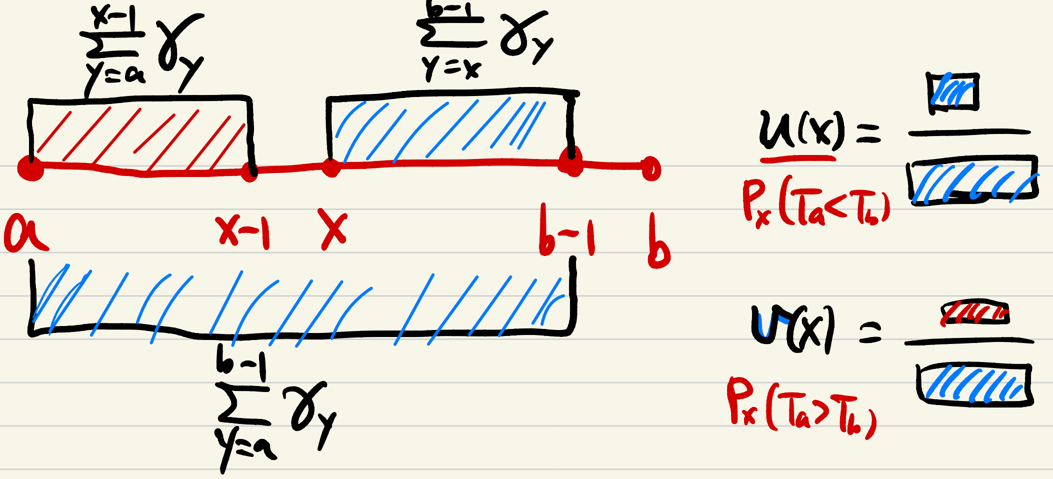
$u(a)=1$ (at $x=a$), $u(b)=0$ (at $x=b$)

Recall: $\gamma_x = \begin{cases} \left(\frac{q_1}{p_1}\right)\cdots\left(\frac{q_x}{p_x}\right) & x=1,2,\cdots \\ 1 & x=0 \end{cases}$



$$\therefore \quad v(x) = 1-\underbrace{u(x)}_{=P_x(T_a<T_b)} = \frac{\sum_{y=a}^{x-1}\gamma_y}{\sum_{y=a}^{b-1}\gamma_y}$$

$v(x) = P_x(T_a>T_b)$



e.g. Recall

- \$1 game with the house
- winning prob $= p = \frac{9}{19}$

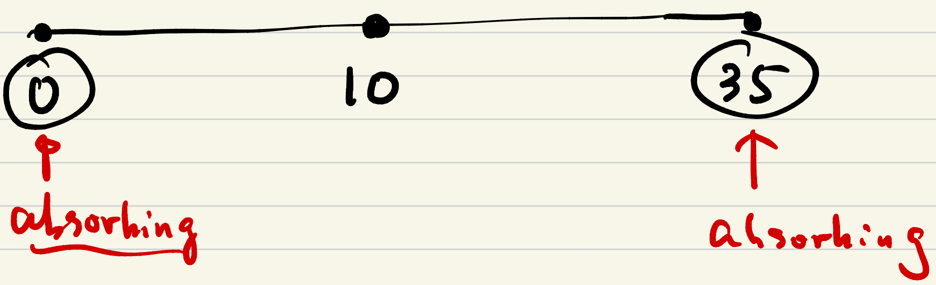
  losing prob $= q = \frac{10}{19}$
- quit the game if

  winning \$25 or losing \$10.

Q: Find prob that he quits the game with winning.

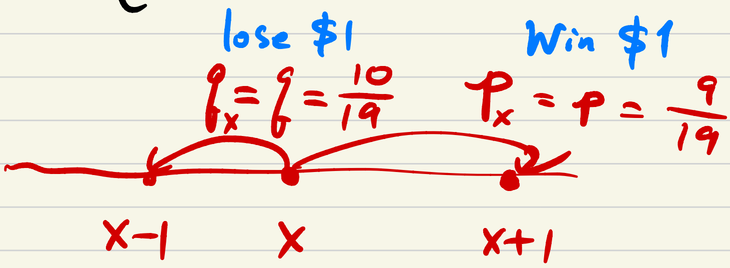
(he wins \$25)

setup : $X_0 = 10$ (w.l.g.)



MC : $X_0 = 10,\ X_1,\ X_2,\ \cdots$

$$S = \{0, 1, \dots, 35\}$$



$$\frac{q}{p} = \frac{10}{9}$$

Convention :

$$q_0 = 0, \quad p_{35} = 0$$

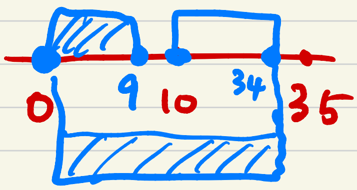
$$\gamma_x = \begin{cases} \underbrace{\overbrace{\left(\frac{q_1}{p_1}\right)}^{\frac{q}{p}} \cdots \overbrace{\left(\frac{q_x}{p_x}\right)}^{\frac{q}{p}}}_{x \text{ terms}} = \left(\frac{q}{p}\right)^x = \left(\frac{10}{9}\right)^x & x = 1, 2, \dots \\ 1 & x = 0 \end{cases}$$

to compute :

$$P\Big(\underbrace{T_{35} < T_0}_{\text{quits with winning } \Leftrightarrow} \;\Big|\; \begin{array}{c}\text{chain starts} \\ \text{at } 10 \\ \text{i.e } X_0 = 10\end{array}\Big)$$

$$= P_{10}(T_{35} < T_0) \qquad (\; v(x) = \dots$$

$$= \frac{\sum_{y=0}^{9} \gamma_y}{\sum_{y=0}^{34} \gamma_y}$$



$$= \frac{\sum_{y=0}^{9} \left(\frac{10}{9}\right)^y}{\sum_{y=0}^{34} \left(\frac{10}{9}\right)^y} = \cdots \approx 0.047.$$

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One more question : Two cases he quits the game

1st winning \$25 (+25)	2nd losing \$10 (−10)
0.047	1 − 0.047

Average money for him to quit the game

$$\simeq (0.047) \times (+25) + (1 - 0.047) \times (-10)$$

$$= \cdots$$

$$= -8.36. \quad \#$$

Continue : a general Birth-death MC

Case : $S = \{0, 1, \cdots\}$ infinite states

Assume : irreducible

(for instance, $p_x > 0$, $q_x > 0$)

Question : is this chain recurrent or not ?

(The answer is NOT trivial,

RK : different from the finite-state space case, Indeed, an irreducible MC over the finite-state space must be recurrent. #

Observation: Chain recurrent iff 0 recurrent

($\because$ this is irreducible)

Thm: Chain recurrent iff $\sum_{x=0}^{\infty} \gamma_x = \infty$

$\Leftrightarrow$ 0 recurrent i.e. $\rho_{00} = 1$

(series divergent)

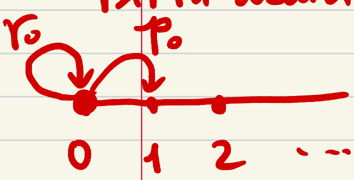
pf: Note:

$$\rho_{00} = P_0 (T_0 < \infty)$$

$$= \underbrace{P(0,0)}_{= r_0} + \sum_{y \neq 0} \underbrace{P(0,y)}_{= \begin{cases} p_0 & y = 1 \\ 0 & y \geq 2 \end{cases}} \rho_{y0} \qquad \sum_{y \neq 0} = \sum_{y=1}^{\infty}$$

$$= r_0 + p_0 \, \rho_{10}$$

Birth-death



$r_0 + p_0 = 1$

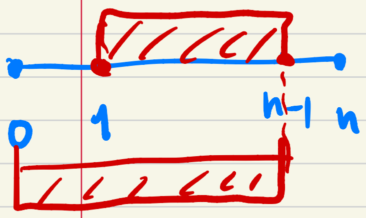
$$\therefore \rho_{00} = 1 \Leftrightarrow \rho_{10} = 1$$

Then, we see:

$$\rho_{10} = P_1 (T_0 < \infty)$$

$$= \lim_{n \to \infty} P_1 (T_0 < T_n)$$

Recall the formula



$$= \lim_{n\to\infty} \frac{\sum_{y=1}^{n-1} \gamma_y}{\left[\sum_{y=0}^{n-1} \gamma_y\right]} \qquad \sum_{y=1}^{n-1} \gamma_y = \sum_{y=0}^{n-1} \gamma_y - \underset{=1}{\gamma_0}$$

$$= 1 - \frac{1}{\sum_{y=0}^{n-1} \gamma_y}$$

$$\therefore \underbrace{\rho_{y0} = 1}_{\Updownarrow} \iff \sum_{y=0}^{\infty} \gamma_y = \infty$$

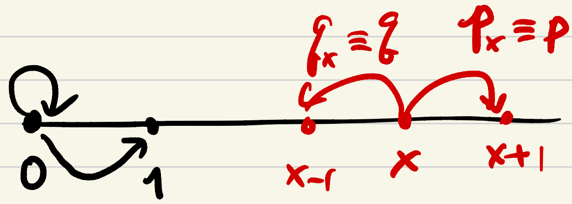
$\rho_{00} = 1$, i.e. 0 is recurrent.

#

RK: In case, birth-death MC

$$S = \{0, 1, \cdots\}$$

$$p_x \equiv p > 0, \quad x = 1, \cdots$$
$$q_x \equiv q > 0$$



$$\gamma_x = \left(\frac{q_1}{p_1}\right) \cdots \left(\frac{q_x}{p_x}\right) \begin{cases} = \left(\frac{q}{p}\right)^x, & x = 1, \cdots \\ = 1 & x = 0 \end{cases}$$

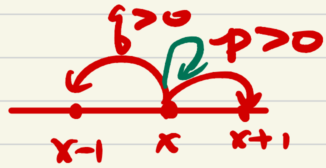
$$q_0 = 0, \quad p_0 = p > 0$$

$\therefore$ irreducible BD MC.

this specific BD MC recurrent $\iff \sum_{x=0}^{\infty} \gamma_x = \sum_{x=0}^{\infty} \left(\frac{q}{p}\right)^x = \infty$

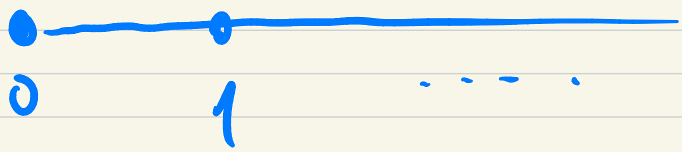
$\iff \frac{q}{p} \geq 1$, i.e. $q \geq p$.

Conclusion:

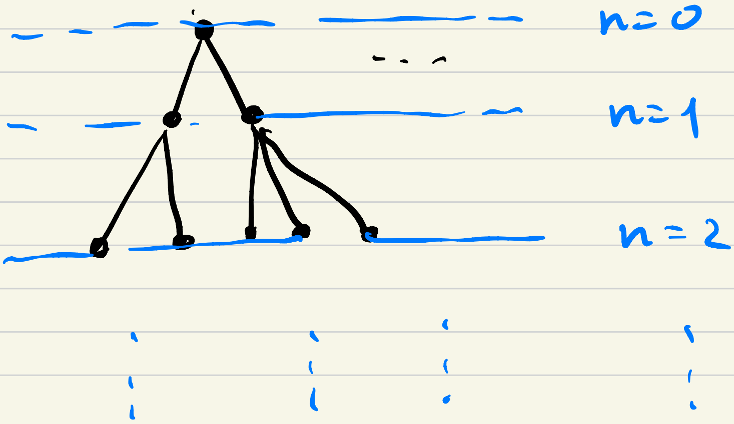


chain $\begin{cases} \text{Recurrent} \iff q \geq p \\ \text{Transient} \iff q < p \end{cases}$

$q \geq p$



Type #2: Branching chain



setting:

each individual generates indep'ly ξ offsprings in the next generation

$X_n \overset{\text{def.}}{=}$ total no of offsprings at the n^{th} generation.

Recall: Transition prob.

$$P(0,0) = 1, \qquad 0 \text{ is absorbing}$$

$$P(x,y) \overset{x \geq 1}{=\!=} P(\underbrace{X_1}_{\xi_1 + \cdots + \xi_x} = y \mid X_0 = x)$$

$$= P(\xi_1 + \cdots + \xi_x = y)$$

Want:

Def. $\rho \overset{\text{def.}}{=} \rho_{10} = P_1(T_0 < \infty)$

is prob that the chain from one individual hits 0 in finite time

Called "extinction probability"

Two trivial cases:

let $p_k = P(\xi = k)$, $k = 0, 1, \ldots$ for any ξ_i

pdf of ξ

then

$$P(1, k) = P(\xi_1 = k) = p_k$$

If $p_0 = 0$, population generated by the 1st individual should never become zero,
$\underset{P(1,0)}{\overset{\parallel}{}}$

$\therefore \quad \rho = 0$

If $p_0 = 1$ .... should extinct for sure,
$\underset{P(1,0)}{\overset{\parallel}{}}$

$\therefore \quad \rho = 1.$

WLG. <u>Assume</u> $0 < p_0 < 1$

(avoid the above two trivial cases)

Let $\mu \overset{\text{def.}}{=\!=} E(\xi) = \sum_{k=0}^{\infty} k\, p_k$

Mean of the r.v. ξ

$$\boxed{\text{Prop. If } \mu < 1 \text{ then } \rho = 1}$$

Pf. : $\rho = \rho_{10}$

$= P_1 (T_0 < \infty)$

$= \lim_{n\to\infty} P_1 (T_0 \leq n)$

$= 1 - \lim_{n\to\infty} P_1 (T_0 > n)$

$\{X_n=0\} \subset \{T_0 \le n\}$

$\Downarrow$

$\underset{\{X_n \ge 1\}}{\underset{\parallel}{\{X_n=0\}^c}} \supset \underset{\{T_0>n\}}{\underset{\parallel}{\{T_0 \le n\}^c}}$

0 1 2

<u>Claim #1°:</u>

$\{T_0>n\} \subset \{X_n \ge 1\}$

$$P_1(T_0>n) \le P_1(X_n \ge 1)$$

$\bigcup_{k=1}^{\infty}\{X_n=k\} = \{X_n \ge 1\}$

$$= \sum_{k=1}^{\infty} 1 \cdot P_1(X_n=k)$$

$k \ge 1$

$$\le \sum_{k=1}^{\infty} k\, P_1(X_n=k)$$

$$= \sum_{k=0}^{\infty} k\, P_1(X_n=k)$$

$$= E(X_n)$$

<u>Claim #2°:</u>

$$E(X_n) = \mu^n E(X_0)$$

If you agree, then claim #1°

$$0 \le P_1(T_0>n) \le E(X_n) = \mu^n E(X_0) \xrightarrow[n\to\infty]{\because \mu<1} 0$$

$$\therefore \lim_{n\to\infty} P_1(T_0>n) = 0$$

$$\therefore \rho = 1 - \lim_{n\to\infty} P_1(T_0>n)$$

$$= 1-0 = 1. \quad \#$$