Coming lectures to discuss three computational issues:

1st issue: Find pdf of X_n, $n \geq 1$.

2nd issue: Find $P(X_n = y \text{ for some } n \geq 1 \mid X_0 = x)$.

3rd issue: $N(y) \stackrel{\text{def.}}{=}$ no. of times the chain X_n $(n \geq 1)$ visits y.

find pdf of $N(y)$.

1st issue: Find pdf of X_n, $n = 1, 2, \cdots$

setup: • $\{X_n\}_{n=0}^{\infty}$: MC (time-homogeneous)

• $S = \{0, 1, \cdots, N\}$

N: finite or ∞

• $P = [P(x,y)]_{x,y \in S} = [P(X_{n+1} = y \mid X_n = x)]$

goal: Find the prob. row vector

$$\pi^{(n)} = [P(X_n = 0), \cdots, P(X_n = N)]$$

pdf of X_n, $n = 1, 2, \cdots$

$$\pi^{(0)} = [P(X_0 = 0), \cdots, P(X_0 = N)]$$ — initial distribution

Known

$\{0, 1, \cdots, N\}$

k^{th} component of $\pi^{(n)}$:

$$= P(X_n = k)$$

$$\Omega = \bigcup_{j \in S} \{X_{n-1} = j\}$$

$$= \sum_{j \in S} P(X_n = k \mid X_{n-1} = j)\, P(X_{n-1} = j)$$

$$= \underbrace{[P(X_{n-1}=0),\ P(X_{n-1}=1), \cdots,\ P(X_{n-1}=N)]}_{=\pi^{(n-1)}} \underbrace{\begin{bmatrix} P(X_n=k \mid X_{n-1}=0) \\ P(X_n=k \mid X_{n-1}=1) \\ \vdots \\ P(X_n=k \mid X_{n-1}=N) \end{bmatrix}}$$

k^{th} Column of P

$k = 0, 1, \ldots, N$

$$\therefore\ \pi^{(n)} = \pi^{(n-1)} P, \qquad n = 1, 2, \cdots$$

pdf of X_n

$$= (\pi^{(n-2)} \cdot P) \cdot P$$

$$= \pi^{(n-2)} P^2$$

square of the Markov matrix P

$$= \cdots$$

$$= \pi^{(0)} P^n$$

nth power of Markov matrix P

pdf of X_0

$$P^n = \underbrace{P \cdot P \cdot \ldots \cdot P}_{n \text{ terms}}$$

$$P^2(x,y) = \sum_{x_1 \in S} P(x,\cancel{x_1})P(\cancel{x_1},y)$$

(x,y)-entry of the matrix P^2

$P^3, \; P^4, \; \dots$

$$P^n(x,y) = \sum_{x_1,x_2,\dots,x_{n-1}\in S} P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y)$$

(x,y)-entry of the matrix P^n

Prop. $P^n(x,y) = P(X_n = y \mid X_0 = x)$

pf: $P(X_n = y \mid X_0 = x)$

$$= P(X_n = y, X_{n-1}\in S, \dots, X_1 \in S \mid X_0 = x)$$

$$= \sum_{\substack{x_{n-1}\in S \\ \vdots \\ x_1 \in S}} P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1 \mid X_0 = x)$$

$P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

$$= \sum_{x_1,\dots,x_{n-1}\in S} \frac{P(X_n = y, X_{n-1} = x_{n-1}, \dots, X_1 = x_1, X_0 = x)}{P(X_0 = x)}$$

$\sum_{x_1\in S}\sum_{x_2\in S}\cdots\sum_{x_{n-1}\in S}$

Claim: numerator $= P(X_0 = x)P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y)$

Assume so, plug it

$$\text{RHS} = \sum_{x_1,\cdots,x_{n-1}\in S} \frac{\cancel{P(X_0=x)}\,P(x,x_1)\cdots P(x_{n-1},y)}{\cancel{P(X_0=x)}}$$

$$= \sum_{x_1,\cdots,x_{n-1}\in S} \underbrace{P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y)}_{n \text{ term}}$$

$$= P^n(x,y). \quad \#$$

Pf of claim:

$$P(\underbrace{X_0=x, X_1=x_1, \cdots, X_{n-1}=x_{n-1}}_{A}, \underbrace{X_n=y}_{B})$$

$P(A\cap B)$
$=P(B|A)P(A)$

$$= P(X_n=y\,|\,A)\,P(A)$$

$$= \underbrace{P(X_n=y\,|\,X_{n-1}=x_{n-1})}_{=P(x_{n-1},y)}\,P(X_0=x,\cdots,X_{n-1}=x_{n-1})$$

$$= \cdots$$

$$= P(x_{n-1},y)P(x_{n-2},x_{n-1})\cdots P(x,x_1)P(X_0=x)$$

$$= P(X_0=x)P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},y).$$

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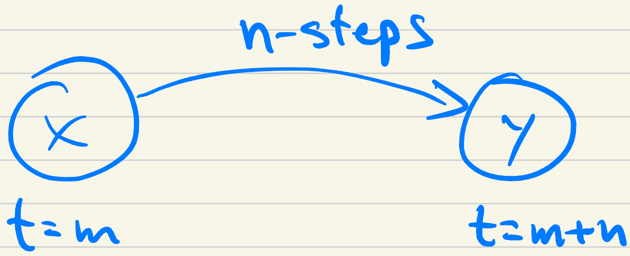
What's more:

Prop. $P^n(x,y) = P(X_{m+n}=y \mid X_m = x)$

$m = 0, 1, 2, \dots$



Pf. RHS

$$= \sum_{\substack{x_{m+n-1}\in S \\ \vdots \\ x_{m+1}\in S}} P(X_{m+n}=y,\ X_{m+n-1}=x_{m+n-1},\ \dots,\ X_{m+1}=x_{m+1} \mid X_0=x_0,\ X_1=x_1,\ \dots,\ X_{m-1}=x_{m-1},\ \underbrace{X_m=x}_{\text{"present"}})$$

$$= \sum_{\substack{x_j\in S \\ j=m+1,\dots,m+n-1}} \frac{P(X_0=x_0,\dots,X_m=x,\ X_{m+1}=x_{m+1},\dots,X_{m+n}=y)}{P(X_0=x_0,\ \dots,\ X_m=x)}$$

= use the previous claim for both numerator and denominator

$$= \sum_{\substack{x_j\in S \\ j=m+1,\dots,m+n-1}} \underbrace{P(x,x_{m+1})P(x_{m+1},x_{m+2})\cdots P(x_{m+n-1},y)}_{n \text{ terms}}$$

$$= P^n(x, y). \quad \#$$

it is reasonable to define:

Def. For $n = 1, 2, \cdots$

$$P^n(x, y) \quad \left(= P(X_n = y \mid X_0 = x)\right)$$

is called the n-step transition function, and P^n called the n-step transition matrix.

Remark:

1° P^n is also a Markov matrix

2° Convention:

$$P^0(x, y) = \delta_{xy} \overset{\text{def.}}{=} \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$P^0 = I$: identity matrix. #

$$P(X_n=y)=\sum_{x\in S}\underbrace{P(X_n=y\mid X_0=x)}_{\substack{=\text{ n-step transition prob.}\\ \text{from } x \text{ to } y\\ = p^n(x,y)}}P(X_0=x)$$

2nd issue: Compute

$$\rho_{xy} \overset{\text{def.}}{=} P(X_n=y \text{ for some } n\geq 1 \mid X_0=x)$$

$x,y\in S$ ← State space

$$= P\left(\left\{\begin{array}{l}\text{The chain } X_n \text{ visits } y\\ \text{in finite time}\end{array}\right\} \middle| X_0=x\right)$$

or

$$\left\{\begin{array}{l}\text{the chain } X_n\ (n\geq 1)\\ \text{ever visit } y\end{array}\right\}$$

$$= P_x\left(X_n\ (n\geq 1) \text{ visits } y \text{ in finite time}\right)$$

Remark:

$$P_x(\cdot)=P(\cdot \mid X_0=x)$$

Def. (Hitting time)

$A \subseteq S$. The hitting time of A is defined by

$$T_A \overset{\text{def.}}{=} \min\{ n \geq 1 : X_n \in A \}$$

i.e. the 1^{st} positive time the chain hits A.

R/K: 1° T_A : a random variable valued in $\{1, 2, 3, \cdots\} \cup \{\infty\}$

Convention:

$$\{T_A = \infty\} = \{ X_n \notin A, \text{ for any } n \geq 1 \}$$
$$= \{ \text{The chain } X_n \ (n \geq 1) \text{ never hits } A \}$$

2°. For $m = 1, 2, \cdots$

$$\{ \underset{\min\{k \geq 1 : X_k \in A\}}{\underset{\parallel}{T_A}} = m \} = \{ X_1 \notin A, \cdots, X_{m-1} \notin A, X_m \in A \}$$

More notations when $A=\{y\}$:

we simply write

$$T_A = T_y = \min\{n\geq 1 : X_n = y\}$$

i.e. the 1^{st} positive time chain visits y.

$$\rho_{xy} = P_x(\underbrace{\text{the chain } X_n\ (n\geq 1) \text{ ever visits } y}_{\text{i.e. } X_n = y \text{ for some } n\geq 1})$$

$$= P_x(T_y < \infty)$$

goal: compute $[\rho_{xy}]_{x,y\in S}$

Prop. (i) $P_x(T_y=1) = P(x,y)$

Compute $P_x(T_y=m)$ $m=1,2,\cdots$ in an iterative way

pf. consequence of

$$\{T_y=1\} = \{X_1=y\}$$

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$$P_x(X_1=y) = P(X_1=y \mid X_0=x)$$
$$= P(x,y). \quad \#$$

(ii) $P_x(T_y=n+1) = \sum_{\text{all } z \text{ in } S} P(x,z)\, P_z(T_y=n)$

but $z \neq y$

for $n = 1, 2, \cdots$

pf. $n = 1, 2, \cdots$

$$\{T_y = n+1\} = \{X_1 \neq y, \cdots, X_n \neq y, X_{n+1} = y\}$$

$$= \bigcup_{\substack{z \in S \\ z \neq y}} \{X_1 = z, X_2 \neq y, \cdots, X_n \neq y, X_{n+1} = y\}$$

disjoint union

$$P_x(T_y = n+1) = \sum_{\substack{z \in S \\ z \neq y}} P_x(\underbrace{X_1 = z}, \underbrace{X_2 \neq y, \cdots, X_n \neq y, X_{n+1} = y})$$

$$= \sum_{\substack{z \in S \\ z \neq y}} \boxed{P_x(X_2 \neq y, \cdots, X_n \neq y, X_{n+1} = y \mid X_1 = z)} \times \underbrace{P_x(X_1 = z)}$$

$$= P(X_1 = z \mid X_0 = x)$$

$$= P(x, z)$$

Markov property

$$= P(X_2 \neq y, \cdots, X_n \neq y, X_{n+1} = y \mid X_0 = x, X_1 = z)$$

$$= P(X_1 \neq y, \cdots, X_{n-1} \neq y, X_n = y \mid X_0 = z)$$

$$= P_z(T_y = n) \quad \#$$

(iii) $$P^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y,y)$$

$n = 1, 2, \cdots$

Pf : $P^n(x,y) = P_x(X_n = y)$

$$\{X_n = y\} \subset \{T_y \le n\}$$

$$\therefore \{X_n = y\} = \{X_n = y\} \cap \underbrace{\{T_y \le n\}}_{= \bigcup_{m=1}^{n} \{T_y = m\}}$$

disjoint union

$$= \bigcup_{m=1}^{n} \{X_n = y, T_y = m\}$$

$P_x(X_n = y)$

$$= \sum_{m=1}^{n} P_x(X_n = y, T_y = m)$$

$$= \sum_{m=1}^{n} \underbrace{P_x(X_n = y \mid T_y = m)}_{1^{st}\ \text{term}} P_x(T_y = m)$$

$\{X_1 \ne y, \cdots, X_{m-1} \ne y, X_m = y\}$

$P(X_n = y \mid \cancel{X_0 = x}, \cancel{X_1 \ne y}, \cdots, \cancel{X_{m-1} \ne y}, X_m = y)$

($n \ge m$

Markov property

$P(X_n = y \mid X_m = y)$

$$= P^{n-m}(y,y). \quad \#$$

Corollary: If $a \in S$ is absorbing
$(\text{i.e. } P(a,a)=1)$

then for $n \geq 1$,

$$P^n(x,a) = P_x(T_a \leq n)$$

pf: $P^n(x,a) = \sum_{m=1}^{n} P_x(T_a = m) \underbrace{P^{n-m}(a,a)}_{\text{claim } 1}$

$$= \sum_{m=1}^{n} P_x(T_a = m)$$

$$= P_x\Big(\bigcup_{m=1}^{n} \{T_a = m\}\Big)$$

disjoint union

$$= P_x(1 \leq T_a \leq n) \quad \#$$

Claim: If a is absorbing,

$$P^n(a,a) = 1, \text{ for any } n = 0, 1, 2, \ldots$$

pf. $n=0$: $P^0(a,a) = 1$

$n=1$: $P^1(a,a) = P(a,a) \overset{\text{by def}}{=} 1$

$n \geq 2$: $P^n(a,a) = \sum_{\substack{x_1 \in S \\ \vdots \\ x_{n-1} \in S}} P(a,x_1)P(x_1,x_2)\cdots P(x_{n-1},a)$

$$= \sum_{\substack{x_2 \in S \\ \vdots \\ x_{n-1} \in S}} \sum_{x_1 \in S} [\;\;]$$

note :

$$P(a, x_1)\, P(x_1, x_2) = \begin{cases} \text{if } x_1 = a : & \overbrace{P(a,a)}^{1} P(a, x_2) = P(a, x_2) \\ \text{if } x_1 \neq a : & = 0 \end{cases}$$

$\Downarrow$ $P(a, x_1) = 0$

$$P^n(a,a) = \sum_{\substack{x_2 \in S \\ \vdots \\ x_{n-1} \in S}} P(a, x_2) \cdots P(x_{n-1}, a)$$

$$= \cdots \quad (\text{ repeat the same argument })$$

$$= P(a,a) = 1. \quad \#$$

Consider

$$\rho_{xx} = P_x (T_x < \infty)$$

prob. that the chain from x returns back to x in finite time

Def. If $\rho_{xx} = 1$, x called "recurrent".

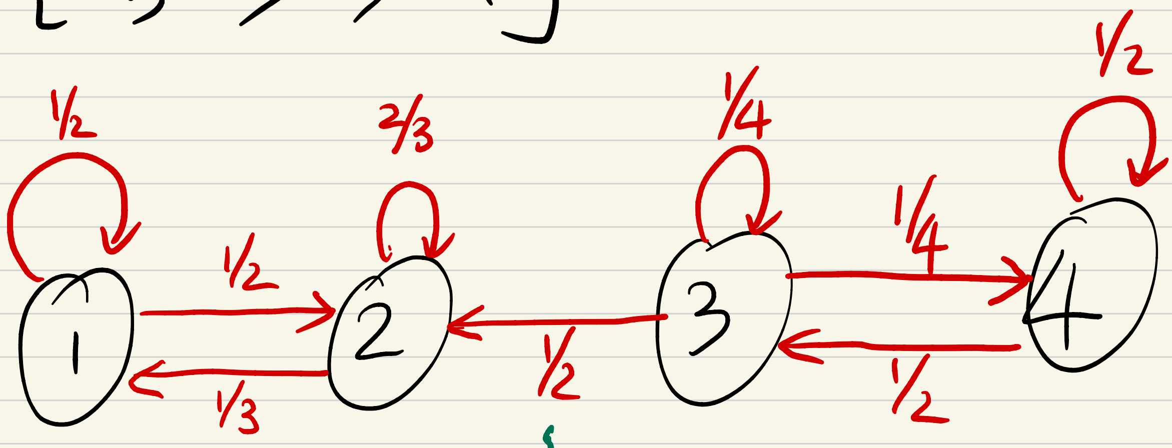
If $\rho_{xx} < 1$, x called "transient".

example: ① Find recurrent / transient states ② Find $[\rho_{xy}]_{x,y\in S}$ in matrix form

Transition matrix $\rightarrow$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

$S = \{1, 2, 3, 4\}$



Observation:

No transition from ② to ③

Recall

Recurrent: $\rho_{xx} = 1$, i.e. $P_x(T_x < \infty) = 1$

$\Updownarrow$

$P_x(T_x = \infty) = 0$

Transient: $\rho_{xx} < 1$, i.e. $P_x(T_x < \infty) < 1$

$$\Updownarrow$$

$$1 - P_x(T_x < \infty) = P_x(T_x = \infty) > 0$$

Claim:

$$\rho_{11} = 1, \quad \rho_{22} = 1, \quad \rho_{33} < 1, \quad \rho_{44} < 1$$

$\therefore$ Recurrent: 1, 2

Transient: 3, 4

In fact,

$\underbrace{\rho_{11}}_{\substack{\| \\ P_1(T_1 < \infty)}} = 1$. Why?

$P_1(T_1 < \infty) \Leftrightarrow P_1(T_1 = \infty) = 0$

otherwise $P_1(T_1 = \infty) > 0$

$\Rightarrow P(2,2) = 1$

2: absorbing

contradiction to "$P(2,1) = \frac{1}{3} > 0$"

Moreover, one can see:

$\underbrace{\rho_{xy}}_{\substack{\| \\ P_x(T_y < \infty)}} = 0$, $\quad x \in \{1, 2\}$, $\quad y \in \{3, 4\}$

for instance, $\rho_{23} = 0$ $\quad \boxed{\because P_2(T_3 < \infty) = 0}$

Now,

$$[\rho_{xy}]_{x,y\in S} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & * & 0 & 0 \\ 2 & * & 1 & 0 & 0 \\ 3 & * & * & \underbrace{*}_{\rho_{33}} & \underbrace{*}_{\rho_{34}} \\ 4 & * & * & \underbrace{*}_{\rho_{43}} & \underbrace{*}_{\rho_{44}} \end{array}$$

how to find "$*$"s ?

We have a general formula to compute each entry:

Claim: $$\underbrace{\rho_{xy}}_{= P_x(T_y<\infty)} \overset{\text{Two-steps idea}}{=} \underbrace{P(x,y)}_{\text{one-step transition}} + \underbrace{\sum_{\substack{z\in S \\ z\neq y}} P(x,z)\rho_{zy}}_{\text{Two-steps transition}}$$

pf: See excerses.

Hint:

$$\{1\le T_y<\infty\} = \underbrace{\{T_y=1\}}_{(\mathrm{I})} \overset{\text{disjoint union}}{+} \underbrace{\{2\le T_y<\infty\}}_{(\mathrm{II})}$$

$$P(\mathrm{I}) = P(x,y)$$

$$(\mathrm{II}) = \{X_1\neq y,\ X_n=y \text{ for some } n\ge 2\}$$

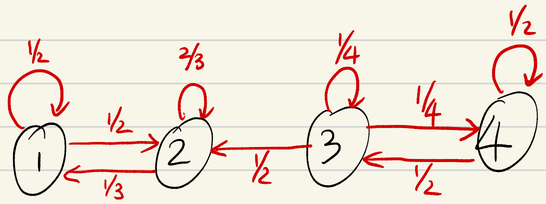
$$= \bigcup_{\substack{z\in S \\ z\neq y}} \{X_1=z,\ X_n=y \text{ for some } n\ge 2\}$$

$$P_x(\mathrm{II}) = \sum_{\substack{z\in S \\ z\neq y}} P_x\big(\underbrace{X_1=z}_{A},\ \underbrace{X_n=y \text{ for some } n\geq 2}_{B}\big)$$

$$\boxed{\begin{aligned} &P_x(A\cap B) \\ &= P_x(B|A)P_x(A) \end{aligned}}$$

$$= \sum_{\substack{z\in S \\ z\neq y}} \underbrace{P_x\big(X_n=y \text{ for some } n\geq 2 \mid X_1=z\big)}_{\rho_{zy} = P_z(1\leq T_y<\infty)} \times \underbrace{P_x(X_1=z)}_{=P(x,z)}$$

for example,



want to compute the 1^{st} column of $[\rho_{xy}]$

$\rho_{11}, \rho_{21}, \rho_{31}, \rho_{41}$

By formula,

$$\rho_{11} \overset{\text{two-steps}}{=} \frac{1}{2} + \frac{1}{2}\rho_{21}$$

$$\rho_{21} = P(2,1) + \underbrace{\sum_{\substack{z\in S \\ z\neq 1}} P(2,z)\rho_{z1}}_{\text{three terms}}$$

$$= \frac{1}{3} + \frac{2}{3}\rho_{21}$$

$$\Rightarrow \rho_{21} = 1, \quad \rho_{11} = 1$$

$$\rho_{31} = 0 + \frac{1}{2}\underbrace{\rho_{21}}_{=1} + \frac{1}{4}\rho_{31} + \frac{1}{4}\rho_{41}$$

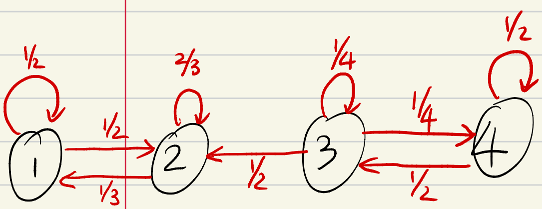
$$\rho_{41} = 0 + \frac{1}{2}\rho_{31} + \frac{1}{2}\rho_{41}$$

$$\Longrightarrow \quad \begin{aligned} \rho_{31} &= 1 \\ \rho_{41} &= 1 \end{aligned}$$

Exercise: Compute $\begin{cases} \rho_{33} \\ \rho_{43} \end{cases}$ and $\begin{cases} \rho_{34} \\ \rho_{44} \end{cases}$?

for instance,

$$\begin{cases} \rho_{33} = \frac{1}{4} + \cancel{\frac{1}{2}\rho_{23}} + \frac{1}{4}\rho_{43} \\ \rho_{43} = \frac{1}{2} + \frac{1}{2}\rho_{43} \Rightarrow \rho_{43} = 1 \end{cases}$$



$\rho_{23} = 0$

$$\therefore \quad \rho_{33} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Sum:

$$[\rho_{ij}] = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 1 & 1 & \frac{2}{3} \end{bmatrix} \end{array}.$$