March 27 (Sat.) :

§3. Basic properties of a general MJP

goal : Given $\{X_t\}_{t\geq 0}$: MJP

how to determine

$$P_{xy}(t) \overset{\text{def.}}{=} P(X_t = y \mid X_0 = x)$$

$$= P(X_{s+t} = y \mid X_s = x)$$

transition function,

prob. that process changes from x to y by taking time t.

Matrix form :

$$P(t) = [\, P_{xy}(t) \,]_{x,y\in S}$$

Prop. (Chapman-Kolmogorov eqn)

$$P_{xy}(t+s) = \sum_{z\in S} P_{xz}(t)\, P_{zy}(s)$$

In matrix form :

$$P(t+s) = P(t)\, P(s)$$

RK: similar to M.C.

$t=m,\ s=n$

$$P^{m+n}(x,y) = \sum_{z\in S} P^m(x,z)\, P^n(z,y)$$

P: Markov matrix $\rightsquigarrow P^{m+n} = \underset{\text{matrix product}}{P^m \cdot P^n}$ #

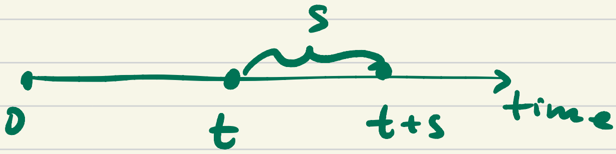
Proof.

$$\text{LHS} = P_{xy}(t+s)$$

$$= P_x(X_{t+s} = y)$$

$$= \sum_{z \in S} P_x(X_{t+s} = y,\ X_t = z)$$

for each $z \in S$.



$$P_x(X_{t+s} = y,\ X_t = z)$$

$P(A \cap B) = P(A|B)P(B)$

$$= P_x(X_{t+s} = y \mid X_t = z)\ P_x(X_t = z)$$

$$= P(X_s = y \mid X_0 = z)\ P_x(X_t = z)$$

$$= P_z(X_s = y)\ P_x(X_t = z)$$

$$= P_{zy}(s)\ P_{xz}(t).$$ #

Rk: Matrix form.

C.-K. means $P(\underbrace{t+s}) = P(t)\,P(s)$

$$\begin{array}{ccc} P(t+s) & = & P(t)\,P(s) \\ \| & & \| \\ P(s+t) & = & P(s)\,P(t) \end{array}$$

$\therefore\ P(t)\,P(s) = P(s)\,P(t),\quad \forall s,\ \forall t.$ #

$P(t),\ t \geq 0$

Now, introduce

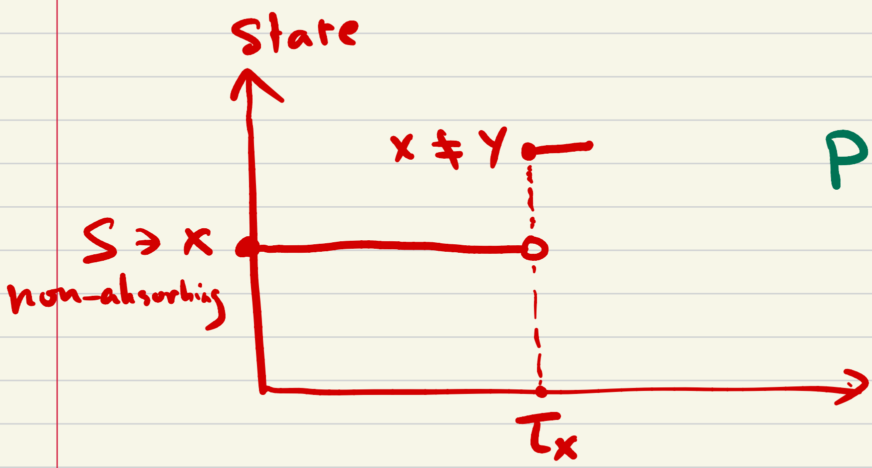
Rate matrix : $D \overset{\text{def.}}{=} P'(0) = [q_{xy}]_{x,y\in S}$

Plan: $\boxed{\text{to show}}$ (two way to dermine $P(t)$)

① $P(t),\ t\geq 0$, (MJP) is uniquely given by D in terms of solving

$$\begin{cases} P'(t) = P(t)\,D = D\,P(t) \\ P(0) = I. \end{cases}$$

② $Q = [Q_{xy}]_{x,y\in S}$ & q_x $(x\in S)$ will be uniquely determined by D, and vice verse.



$$P(T_x > t) = e^{-q_x t},$$

$$q_x = \frac{1}{E(T_x)}$$

↑ rate leaving away from x

(convention : If x absorbing

$E(T_x) = \infty$, $q_x = 0$, $T_x = \infty$)

Coro. Let $D \overset{\text{def.}}{=} P'(0)$ (rate matrix), then

$$P'(t) = P(t)\,D = D\,P(t), \quad \forall t > 0.$$

Proof. C.-K. : $P(t+s) = P(t)\,P(s)$, $\forall t \geq 0$, $\forall s \geq 0$.

$\left.\frac{d}{ds}\right|_{s=0}$ (fix t) $\Rightarrow$ $\left.P'(t+s)\right|_{s=0} = P(t)\left.P'(s)\right|_{s=0}$

$\therefore\ P'(t) = P(t)\,\underbrace{P'(0)}_{=D}$

$\left.\frac{d}{dt}\right|_{t=0}$ (fix s) $\Rightarrow$ $\left.P'(t+s)\cdot 1\right|_{t=0} = \left.P'(t)\,P(s)\right|_{t=0}$

$\therefore\ P'(s) = \underbrace{P'(0)}_{=D}\,P(s)$

#

Question : $\overset{P'(0)}{\overset{\|}{D}}$? where $P(t)$: transition function of a general MJP.

rate matrix

Fact #1 :

$$D = \begin{bmatrix} - & + & + & \cdots & \\ + & - & + & \cdots & \\ + & + & - & & \\ \vdots & \vdots & & \ddots & \end{bmatrix}$$

all diaganol entries "$-$" ≤ 0

other entries "$+$" ≥ 0

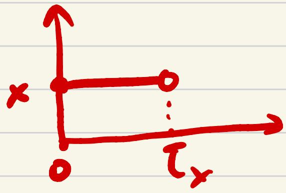
Pf : $D = P'(0) = [q_{xy}]_{x,y\in S}$

$$q_{xy} \overset{\text{def.}}{=} P'_{xy}(0)$$

$$= \lim_{h\to 0+} \frac{P_{xy}(h) - P_{xy}(0)}{h}$$

$$P_{xy}(0) = \begin{cases} 1 & x=y \\ 0 & \text{otherwise} \end{cases}$$

$$= \lim_{h\to 0+} \frac{P_{xy}(h) - \delta_{xy}}{h}$$



$$= \begin{cases} \text{if } x=y & = \lim\limits_{h\to 0+} \dfrac{\overset{\in[0,1]}{P_{xy}(h)} - 1}{\underset{>0}{h}} \quad \left.\leq 0\right\} \leq 0 \\ \text{if } x\neq y & = \lim\limits_{h\to 0+} \dfrac{\overset{\in[0,1]}{P_{xy}(h)} - 0}{\underset{>0}{h}} \quad \left.\geq 0\right\} \geq 0 \end{cases}$$

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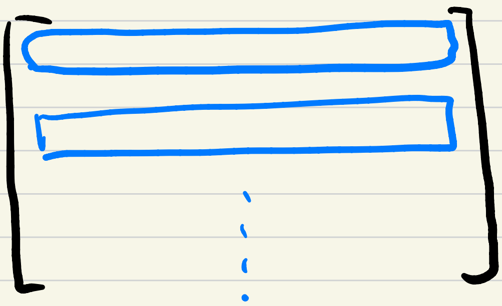
Fact #2.

$D =$ 

each row sum $= 0$

i.e.

$= [q_{xy}]_{x,y\in S}$

$$\sum_{y\in S} q_{xy} = 0$$

$\forall x\in S$

pf:

$$P(t) = [P_{xy}(t)]_{x,y}, \quad \forall t \geq 0$$

Markov

$$\sum_{y \in S} P_{xy}(t) = 1, \qquad \forall t \geq 0.$$

$$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow$$

$$\sum_{y \in S} \underbrace{P'_{xy}(0)}_{= q_{xy}} = 0 \qquad \#$$

Remark. If x is absorbing,

then the row vector associated with x in the rate matrix D must be just zero

x absorbing

$\Downarrow$

$$P(X_t = y \mid X_0 = x) = P_{xy}(t) = \delta_{xy}, \quad \forall t \geq 0$$

$\Downarrow$

$$q_{xy} = P'_{xy}(0) = 0, \quad \forall y \in S$$

x $\boxed{0 \ 0 \ \cdots\cdots}$ in D

absorbing

means: rate of changing away from any absorbing state x to any state y must be zero def. of a general MJP

Recall: $$Q_{xy} \overset{!}{=} \delta_{xy} = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{otherwise} \end{cases}$$

transion prob. changing from x to y

Note: By two facts above,

x: $[\ - - - - - \ q_{xx} \ - - - - - \ ]$ in $D=[q_{xy}]$

$\oplus \quad \oplus \quad <0 \quad \oplus \ - - - \ \oplus$

$$\sum_{y \in S} q_{xy} = 0$$

$$\Updownarrow$$

$$\boxed{\underbrace{-q_{xx}}_{\geq 0} = \sum_{\substack{y \in S \\ y \neq x}} \underbrace{q_{xy}}_{\geq 0}}$$

$D = [q_{xy}] = P'(0)$

rate matrix

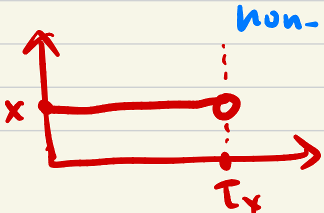
rate of leaving x

rate of changing away from x to y $(\neq x)$

e.g.

x

non-absorbing



guess: $-q_{xx} = q_x = \frac{1}{E(\tau_x)}$

Thm :

$$\begin{cases} -q_{xx} = q_x \\ q_{xy} = q_x Q_{xy}, \quad \forall y \neq x \end{cases}$$

(distribute q_x by Q_{xy})

Rk.

$$\boxed{D = P'(0) = [q_{xy}]} \Longleftrightarrow \boxed{\begin{cases} Q = [Q_{xy}] \\ q_x, \ x \in S \end{cases}}$$

Proof.

Easy case : x is absorbing.

$$E(\tau_x) = \infty$$

$$q_x = \frac{1}{E(\tau_x)} = 0$$

TRUE.

Case : x is non-absorbing ($\tau_x < \infty$)

waiting time to jump

$$P_{xy}(t) = P_x(X_t = y)$$

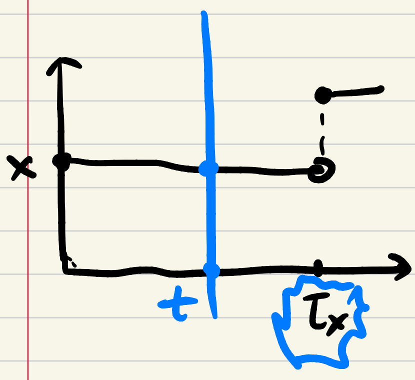
$$= P_x(X_t = y, \ \tau_x > t) \quad \text{— (I)}$$

no jump up to t

$$+ P_x(X_t = y, \ \tau_x \leq t) \quad \text{— (II)}$$

jump occurs before t.

(I) :



$X_s = x,\ 0 \le s \le t$

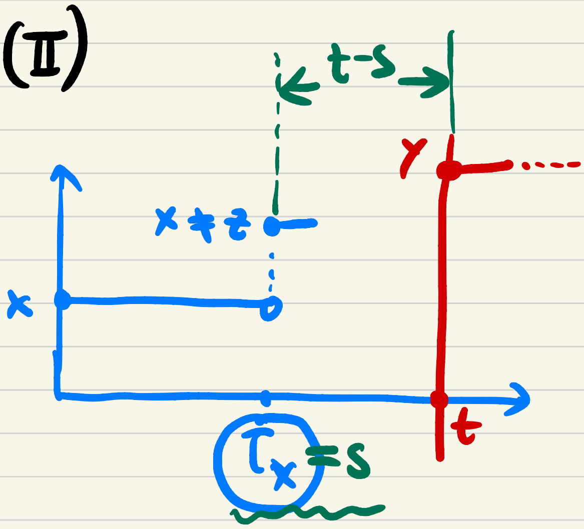
$$(\mathrm{I}) = P_x(X_t = y,\ \tau_x > t)$$

$$= \begin{cases} 0 & \text{if } y \neq x \\ P_x(\tau_x > t) = e^{-q_x t} & \text{if } y = x \end{cases}$$

exp. r.v.

$\{\tau_x > t\} \subset \{X_t = x\}$ $$= e^{-q_x t}\,\delta_{xy}$$

(II)



$$(\mathrm{II}) = P_x(X_t = y,\ \tau_x \le t)$$

$$= \sum_{x \neq z \in S} P_x(\underset{\substack{\| \\ s}}{\tau_x} \le t,\ X(\tau_x) = z,\ X_t = y)$$

$$= \sum_{x \neq z \in S} \int_0^t q_x e^{-q_x s} \cdot Q_{xz} \cdot P_{zy}(t-s)\, ds$$

$t - s = u$

$ds = -du$

Sum :

$$P_{xy}(t) = (\mathrm{I}) + (\mathrm{II})$$

$$= e^{-q_x t}\,\delta_{xy}$$

$$+ \; q_x e^{-q_x t} \sum_{x \neq z \in S} \int_0^t Q_{xz} P_{zy}(u)\, e^{q_x u}\, du$$

(Exercise)

$$\frac{d}{dt} \Rightarrow$$

$$P'_{xy}(t) = -q_x P_{xy}(t)$$

$$+ \; q_x \cancel{e^{-q_x t}} \sum_{x \neq z \in S} Q_{xz} P_{zy}(t) \cancel{e^{q_x t}}$$

$$(\ldots)\Big|_{t=0} \Rightarrow$$

$$q_{xy} = P'_{xy}(0) = -q_x \delta_{xy}$$

$$+ \; q_x \sum_{z \neq x} Q_{xz} \delta_{zy}$$

$$= \begin{cases} \text{if } y = x: & = -q_x \cdot 1 + 0 = -q_x \\ \text{if } y \neq x: & = -q_x \cdot 0 + q_x \cdot Q_{xy} \\ & = q_x Q_{xy}. \end{cases}$$

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Example: Poisson process with arrival rate $= \lambda > 0$

(Satisfies the thm).

Direct computation : $\{X_t\}_{t \geq 0}$, X_0 : r.v.

transition function

$$P_{mn}(t) \overset{t>0}{=\!=} P(X_{s+t} = n \mid X_s = m)$$
$s>0$

$$= \begin{cases} \text{if } n < m, & = 0 \\ \text{if } n \geq m, & = P(X_t = n-m \mid X_0 = 0) \end{cases}$$

Poisson Process $\longrightarrow$ $= P_0(X_t = n-m)$

$$= e^{-\lambda t} \frac{(\lambda t)^{n-m}}{(n-m)!}$$

matrix form :

$$P(t) = \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{bmatrix} e^{-\lambda t} & e^{-\lambda t}\frac{(\lambda t)^1}{1!} & e^{-\lambda t}\frac{(\lambda t)^2}{2!} & \cdots \\ 0 & e^{-\lambda t} & e^{-\lambda t}\frac{(\lambda t)^1}{1!} & \cdots \\ & & \ddots & \ddots \\ & 0 & & \end{bmatrix}$$

$\Rightarrow$ $P(0) = I$, $P_{xy}(0) = \delta_{xy}$

$$D = P'(0) = \left.\frac{d}{dt}\right|_{t=0} P(t)$$

$$= \cdots \qquad (\text{Exercise})$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \end{array} \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & & \\ 0 & -\lambda & \lambda & 0 & \cdots & \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ & & & \ddots & \ddots & \end{bmatrix}$$

rate matrix

From the def of P. P.

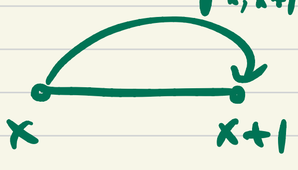
$$Q = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \cdots & \\ \hline 0 & 0 & 1 & 0 & 0 & \cdots & \\ 1 & 0 & 0 & 1 & 0 & \cdots & \\ 2 & 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & & & \ddots & \ddots \end{array}$$

Markov matrix.

Exercise: check : $\forall x \in \{0, 1, \cdots\}$

$$\begin{cases} -q_{xx} = q_x = \lambda, \\ q_{xy} = q_x Q_{xy}, \quad \forall y \neq x. \end{cases}$$

rate $q_{x,x+1} = \lambda Q_{x,x+1}$
$= \lambda$



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