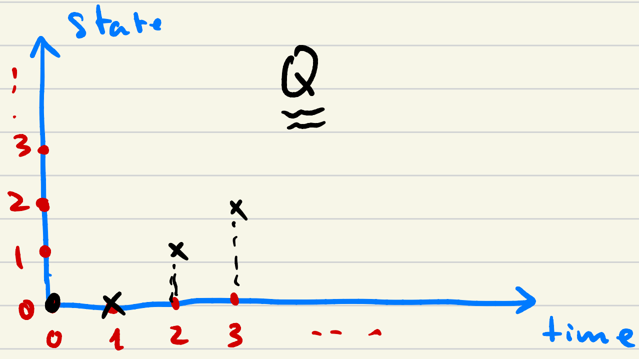
March 22:

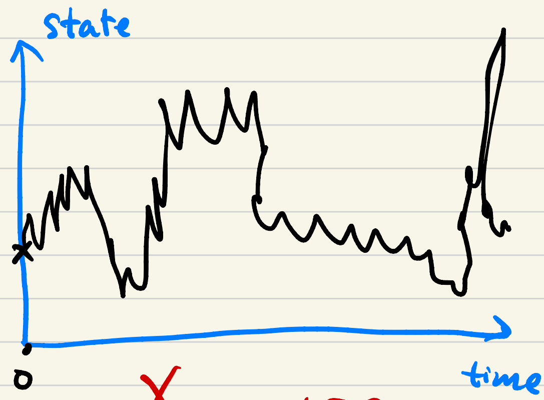
Chap 3. Markov Jump Process

* Discrete-in-time Stochastic Process

* Continuous-in-time Stochastic Process



X_t, $t = 0, 1, 2, \dots$



X_t, $t \geq 0$

§1. What's MJP?

$X(t)$, $0 \leq t < \infty$

Def. A continuous-in-time SP

$$X_t, \quad 0 \leq t < \infty$$

that takes the values in state space S (finite or contably infinite) and is defined on a common prob. space $(\Omega, \mathcal{F}, P)$, is a MJP if

① it is a jump process (explain later);

② Markov property is satisfied:

$$P(X_t = y \mid X_{s_1} = x_1, \dots, X_{s_n} = x_n, X_s = x)$$
$$= P(X_t = y \mid X_s = x)$$

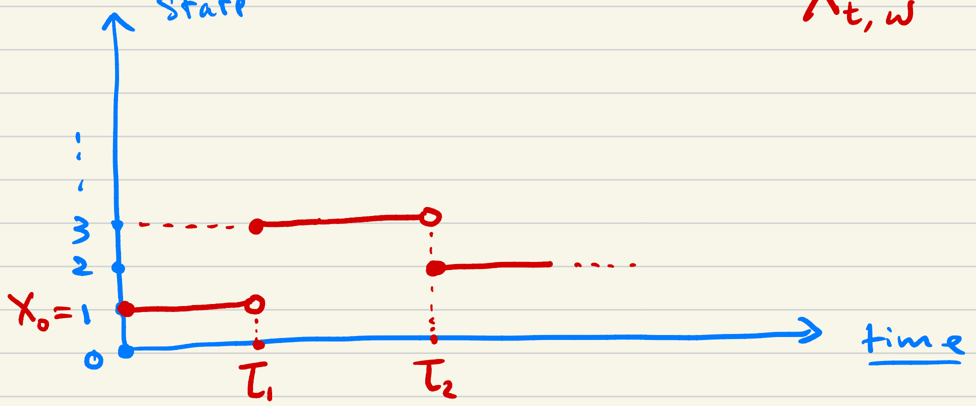
$\forall\ 0 \le s_1 \le s_2 \le \cdots \le s_n < s \le t$

present

$\forall\ x_1, x_2, \cdots, x_n, x, y \in S$

What's a jump process?

$X_{t,\omega}$

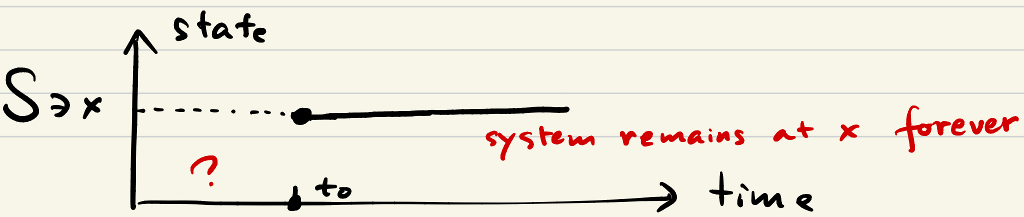


* time to jump : $\tau_1, \tau_2, \cdots$
* where to jump : $X_{\tau_1}, X_{\tau_2}, \cdots$
* no blow-up : $\lim_{n\to\infty} \tau_n = \infty$
* "Waiting time to jump" & "where to jump" } independent (explained later)

Def. $x \in S$ is an absorbing state if

"$X_{t_0} = x$ at some $t_0 \ge 0$"

$\Rightarrow$ "$X_t = x, \ \forall\, t \ge t_0$"



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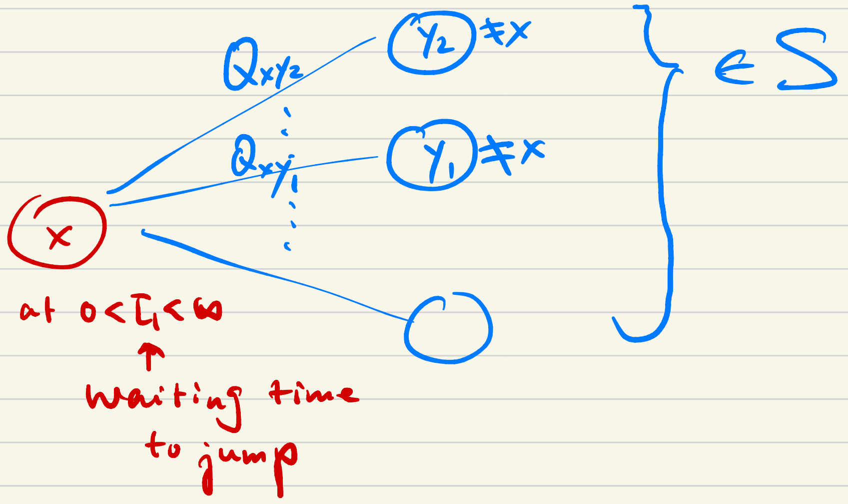
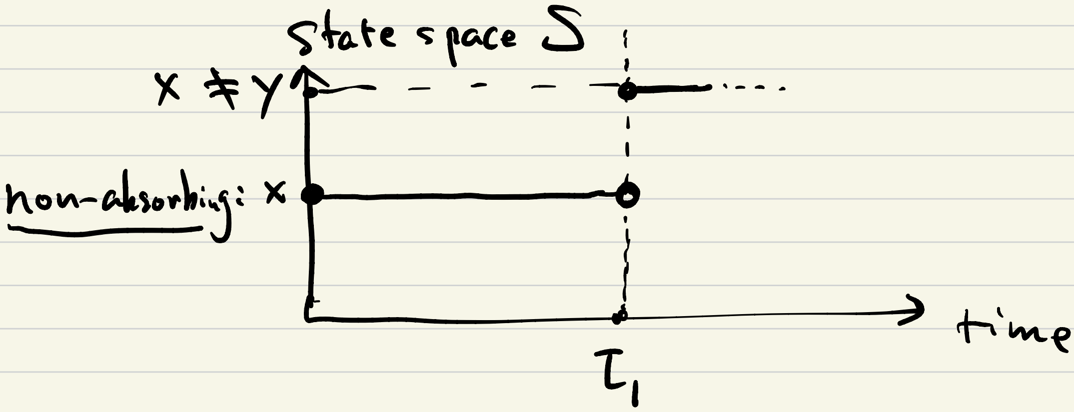
If $X_0 = x$ for a non-absorbing state $x \in S$, then this process must jump from x to another state, for instance, $y\,(\neq x) \in S$





Def.: For x non-absorbing,

$Q_{xy} \overset{y \neq x}{=\!=}$ the transition prob. that the process jumps from x to $y\,(\neq x)$,

non-absorbing

and

$Q_{xx} = 0$. (means: At τ_1, process must make a jump)

Note:

$$\sum_{\substack{y \in S \\ y \neq x}} Q_{xy} = 1$$

#

Convention : If x is absorbing,

$$Q_{xy} = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{xy}.$$

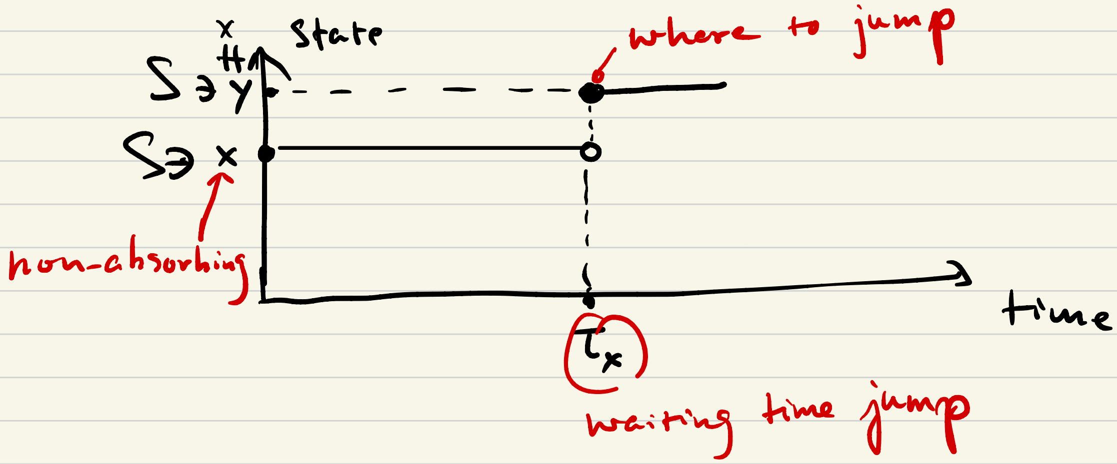
Remark : Now, you get a trasition Matrix

$$Q = [Q_{xy}]_{x,y \in S}$$

Markov matrix

About independence

"waiting time to jump" & "where to jump"



means :

$$P_x(\tau_x \leq t, X(\tau_x) = y)$$

$$= P_x(\tau_x \leq t)\, Q_{xy}$$

Big goal to show:
(non-trivial)

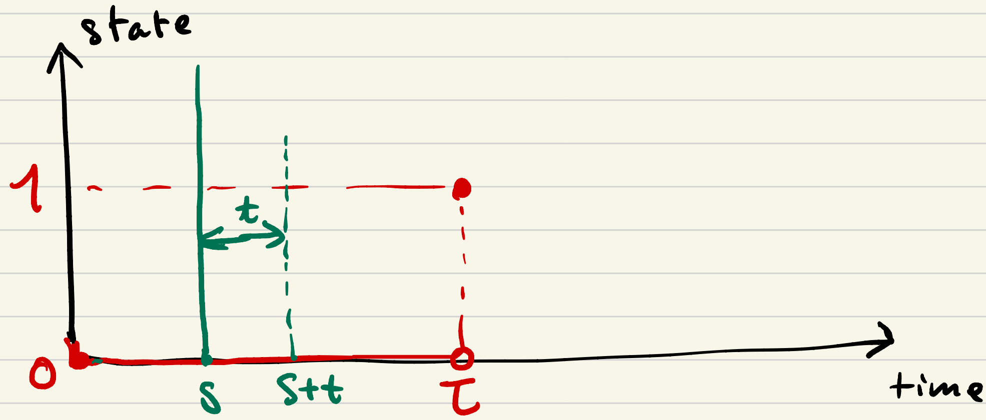
If $\{X_t\}_{t \geq 0}$ is a MJP,

then for a non-absorbing state $x \in S$,

τ_x is an exponential r.v. !!!

Def. A r.v. $\tau \in [0, \infty)$ is memoryless if

$$P(\tau > s+t \mid \tau > s) = P(\tau > t), \quad \forall s, t \geq 0.$$

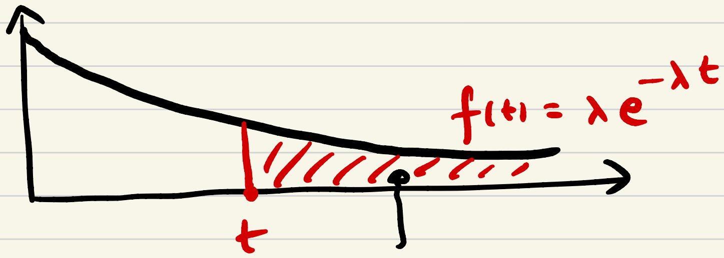


Model: just think of τ as a r.v. denoting the waiting time for a bus to arrive.

Prop. A memoryless r.v. $\tau \in [0, \infty)$ must be exponential with

$$\underbrace{P(\tau > t)}_{\overset{\text{def.}}{=} G(t)} = e^{-\lambda t}, \quad \forall t \geq 0$$

where $\lambda = \dfrac{1}{E(\tau)}$



$$e^{-\lambda t} = P(\tau > t)$$

Proof. $G(t) \overset{t \geq 0}{=} P(\tau > t)$

$$\overset{\text{memoryless property}}{=} P(\tau > \boxed{s+t} \mid \tau > s), \quad \forall s \geq 0$$

$$\overset{\text{def. of cond. prob.}}{=} \frac{P(\tau > s+t,\ \tau > s)}{P(\tau > s)}$$

$$= \frac{P(\tau > s+t) = G(s+t)}{P(\tau > s) = G(s)}$$

i.e. $G(s+t) = G(s)\,G(t), \quad \forall s, t \geq 0.$

Assume: G is differentiable, then

$$G'(t) \overset{t \geq 0}{=} \lim_{h \to 0+} \frac{G(t+h) - G(t)}{h}$$

$$= \lim_{h \to 0+} \frac{G(t)\,G(h) - G(t)}{h}$$

$$= G(t) \underbrace{\lim_{h \to 0+} \frac{G(h) - 1^{= G(0)}}{h}}$$

Note: $G(0) = P(\tau > 0) = P(\Omega) = 1$

$= G'(0) \overset{\text{def.}}{=} \underset{\text{Constant}}{\alpha \in \mathbb{R}}$

$\therefore \quad G'(t) = \alpha G(t), \quad \forall t \geq 0.$

$\therefore \quad G(t) = G(0) e^{\alpha t} = e^{\alpha t}$

Note:

$G(t) = P(\tau > t) \downarrow$ in time

tells: $\underset{\sim}{\alpha < 0}$

let $\alpha = -\lambda$, for $\underset{\sim}{\lambda > 0}$

$\therefore \quad P(\tau > t) = G(t) = e^{-\lambda t}$. #

$\Downarrow$

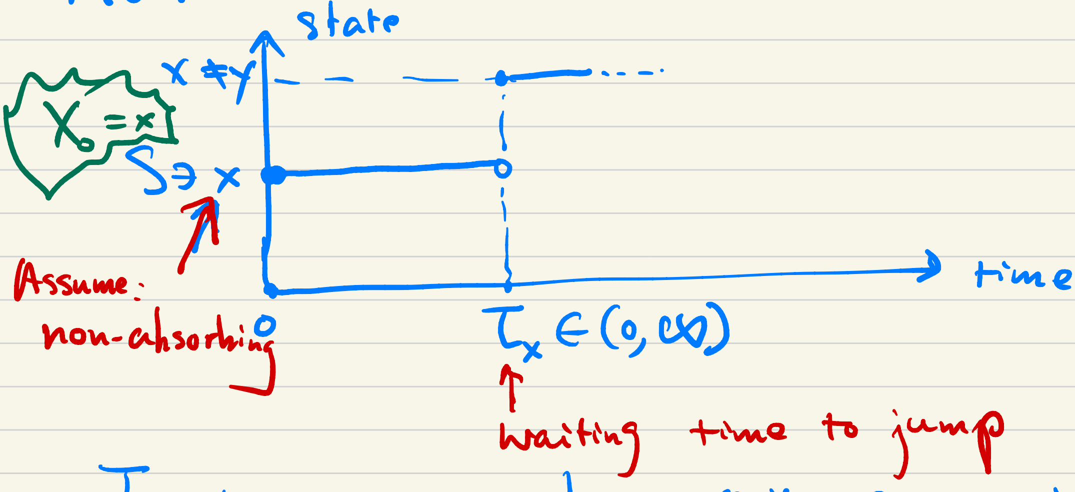
τ is an exponential r.v.

with parameter $\lambda > 0$

Exercise: $E(\tau) = \frac{1}{\lambda}$

$\therefore \quad \lambda \overset{\text{def.}}{=} \frac{1}{E(\tau)}$.

Prop. $\{X_t\}_{t \geq 0}$ is a (time-homogenous) MJP



Then τ_x is a memoryless r.v. so it's exponential.

Proof. $\forall s, r \geq 0$,

$(s+r) - s = r$

$$P(\tau_x > s + r \mid \tau_x > s) \quad (\neq P(\tau_x > r))$$

$$= P(X_t = x, \ \underbrace{0 \leq t \leq s+r}_{\text{on or befor } s+r} \mid X_t = x, \ 0 \leq t \leq s)$$

on or befor s+r process remains at state x

$$= P(X_t = x, \ s < t \leq s+r \mid X_t = x, \ 0 \leq t \leq s)$$

present

Markov property

$$= P(X_t = x, \ s < t \leq s+r \mid X_s = x)$$

time-homogeneous, i.e. regard time s as time 0

$$= P(X_t = x, \ 0 < t \leq r \mid X_0 = x)$$

$$= P_x(X_t = x, \ 0 < t \leq r)$$

$$= P_x(\tau_x > r). \quad \#$$

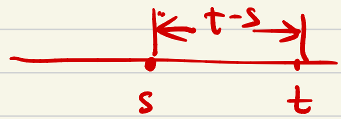
e.g. $$P(\tau_x > t) = e^{-\lambda_x t}$$

$$\lambda_x = \frac{1}{E(\tau_x)} > 0$$

for any x non-absorbing

Remark: We always assume the time-homogeneous property for any MJP in this course, i.e.

$$P(X_t = y \mid X_s = x) \;\underset{}{\overset{0\le s\le t}{=\!=}}\; P(X_{t-s} = y \mid X_0 = x)$$



Def. $\{X_t\}_{t\geq 0}$: MJP (time-homogeneous)

$$P_{xy}(t) \overset{\text{def.}}{=} P_x(X_t = y) = P(X_t = y \mid X_0 = x)$$

$t \geq 0,\ y \in S$ $\quad = P(X_{t+s} = y \mid X_s = x)$

transition function. #