MATH2010 Advanced calculus, 2020-21
HOMEWORK TWO
Suggested Solution

1. (a) We need to find all r such that w satisfies the differential equation. Note that

$$\begin{aligned} w_{xx} &= \frac{1}{c^2} w_t \\ \frac{d}{dx}(\pi e^{rt}\cos \pi x) &= \frac{r}{c^2} e^{rt} \sin \pi x \\ -\pi^2 e^{rt} \sin \pi x &= \frac{r}{c^2} e^{rt} \sin \pi x \end{aligned}$$

Comparing two sides of the equation, we have $r = -(c\pi)^2$.
Obviously the function $w(x,t) = e^{-c^2\pi^2 t}\sin(\pi x)$ satisfies the differential equation.

(b) We need to determine all r and k.
Comparing two sides of the equation, we have $r = -(ck)^2$, similar as in (a).
By $0 = w(L,t) = e^{-c^2k^2t}\sin(kL)$, we have $kL = n\pi$, for $n \in \mathbf{Z}$ as $e^{-c^2k^2t} > 0$.
Hence, it is direct to check that the following is all the solutions in the required form,

$$w(x,t) = e^{-\frac{n^2c^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right), \forall n \in \mathbf{Z}.$$

The solutions will tend to 0 as t tends to ∞.

2. (a) Differentiable, and hence also continuous.
Note that the partial derivatives $\frac{\partial}{\partial x} f(x,y) = \sin y$ and $\frac{\partial}{\partial y} f(x,y) = x\cos y$ exist and are continuous on $\mathbf{R}^2$. Hence f is $\mathcal{C}^1$. It follows that f is differentiable, and hence also continuous.

(b) Continuous but not differentiable.
Since f is a composition of the continuous function xy and the absolute value function, f is continuous. For differentiability, note that $f(x,1) = |x|$, so $\frac{\partial}{\partial x} f(0,1)$ does not exist. Hence, f is not differentiable at $(0,1)$ and so is not a differentiable function on $\mathbf{R}^2$.

(c) Continuous but not differentiable.
Clearly $f(x,y)$ is continuous for $x \neq 0$. Consider any point $(0,y_0)$ on the y-axis. Note that

$$-|xy| \leq |f(x,y)| \leq |xy| \quad \text{and} \quad \lim_{(x,y)\to(0,y_0)} |xy| = 0.$$

By sandwich theorem,

$$\lim_{(x,y)\to(0,y_0)} f(x,y) = 0 = f(0,y_0)$$

Hence, f is also continuous at $(0, y_0)$. For differentiability, note that

$$\frac{\partial}{\partial x} f(0,1) = \lim_{h\to 0} \frac{f(h,1) - f(0,1)}{h} = \lim_{h\to 0} \frac{h \sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} \sin\frac{1}{h}$$

does not exist. Hence, f is not differentiable at $(0,1)$ and so is not a differentiable function on $\mathbf{R}^2$.

(d) Continuous but not differentiable.
Clearly f is continuous for $(x,y) \neq (0,0)$. By using polar coordinates,

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^{\frac{3}{2}}} \\
&= \lim_{r\to 0} \frac{r^4 \cos^2\theta \sin^2\theta}{r^3} \\
&= \lim_{r\to 0} r\cos^2\theta\sin^2\theta \\
&= 0 \qquad \text{(by sandwich theorem)} \\
&= f(0,0)
\end{aligned}$$

Hence, f is continuous on $\mathbf{R}^2$.
We will show $f(x,y)$ is not differentiable at $(0,0)$. Since $f(x,y) = 0$ for $x = 0$ or $y = 0$, we have $\frac{\partial}{\partial x} f(0,0) = \frac{\partial}{\partial y} f(0.0) = 0$.
Hence the linear approximation of f at $(0,0)$ is given by

$$L(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)(x-0) + \frac{\partial f}{\partial y}(0,0)(y-0) = 0.$$

It follows that

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} &= \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{(x^2+y^2)^2} \\
&= \lim_{r\to 0} \frac{r^4\cos^2\theta\sin^2\theta}{r^4} \\
&= \lim_{r\to 0} \cos^2\theta\sin^2\theta
\end{aligned}$$

does not exists as it depends on θ.

3. From the definition and the limit equation of the error ϵ, we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} &= \lim_{h\to 0} \frac{\epsilon(x_0+h,y_0) + r(x_0+h-x_0) + s(y_0-y_0)}{h} \\
&= r + \lim_{h\to 0} \frac{\epsilon(x_0+h,y_0)}{h} = r.
\end{aligned}$$

By the definition of the partial derivative, we have $r = f_x(x_0, y_0)$. Similarly,

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} &= \lim_{h\to 0} \frac{\epsilon(x_0,y_0+h) + r(x_0-x_0) + s(y_0+h-y_0)}{h} \\
&= s + \lim_{h\to 0} \frac{\epsilon(x_0,y_0+h)}{h} = s.
\end{aligned}$$

We have $s = f_y(x_0, y_0)$.