MATH2050A HW3 Solution

9. We prove the statement by contradiction. Suppose that $\lim X = 0$ is false. (Note that we cannot say $\lim X \neq 0$ as the sequence may not even converge). By definition of limit of a sequence, we have $\exists\ \epsilon > 0$ s.t. $\forall\ k \in \mathbb{N}\ \exists\ m > k$ s.t. $|x_m - 0| \geq \epsilon$.

   We now construct a subsequence inductively. Let $n_1 = 1$. For each $k \in \mathbb{N}$, $n_k \in \mathbb{N}$, so $\exists\ n_{k+1} > n_k$ s.t.$\left|x_{n_{k+1}} - 0\right| \geq \epsilon$. Therefore, we obtain a subsequence x_{n_k} of X. But any subsequence $x_{n_{k_j}}$ of x_{n_k} cannot converge to 0 because $\left|x_{n_{k_j}} - 0\right| \geq \epsilon$ for $j > 1$. Contradiction arises. So the original statement must be true.

14. Let X be the set $\{x_n : n \in \mathbb{N}\}$. Since s is the supremum of X, $s - \frac{1}{n}$ cannot be an upper bound of X. Also, since s is not in the set and s is an upper bound of X, any element in X is strictly smaller than s and thus cannot be an upper bound of X. Therefore, for each $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ s.t. $x_{n_i} > \max\{x_k : k \leq n_{i-1}\} \cup \{s - \frac{1}{i}\}$. By construction, the index we pick is always greater than that in previous steps. So $\{n_k\}$ is a strictly increasing sequence of natural number and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. Since $\lim_i\ s - \frac{1}{i} = s$ and $s - \frac{1}{i} < x_i < s$, by squeeze theorem, we have $\lim x_{n_k} = s$.

    Remark. Some of the students did not prove that we can find $n_1 < n_2 < n_3 < \ldots$ satisfying the required property. Be careful that $\{n_k\}$ has to be strictly increasing natural numbers so that $\{x_{n_k}\}$ is a subsequence.

3. (b) Let $\epsilon = 1$. For any natural number H, pick $n = H$ and $m = H + 2$, then $|x_n - x_m| = (H + 2 + \frac{(-1)^{H+2}}{H+2}) - (H + \frac{(-1)^H}{H}) = 2 + (-1)^H(\frac{1}{H+2} - \frac{1}{H}) > 1 = \epsilon$. Therefore, the sequence is not Cauachy sequences.

10. Since $|x_{n+2} - x_{n+1}| = |\frac{1}{2}(x_{n+1} + x_n) - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$, the sequence is contractive and by Theorem 3.5.8, the sequence converges.

    Now we calculate the limit of the sequence. For $n \geq 3$ since

$$\sum_{k=3}^{n} x_k = \frac{1}{2}\sum_{k=2}^{n-1} x_k + \frac{1}{2}\sum_{k=1}^{n-2} x_k = \frac{1}{2}x_1 + \sum_{k=2}^{n-2} x_k + \frac{1}{2}x_{n-1},$$

    We have $x_n + \frac{1}{2}x_{n-1} = \frac{1}{2}x_1 + x_2$.

    Since $\lim x_n$ exist, we let L to be the limit. Then by taking limit in both sides, we have $L + \frac{1}{2}L = \frac{1}{2}x_1 + x_2$, implying $L = \frac{x_1 + 2x_2}{3}$.

    Remark. This solution is given by one of the students, which uses the fact about convergence of the sequence proved in the previous part. My replaced solution is to prove the general form $x_n = \frac{2}{3} - \frac{2}{3}(-\frac{1}{2})^n(x_2 - x_1) + x_1$ by MI. Special thanks to him/her.