MATH2050A HW1 Solution

1. Show that $\sup\{1 - 1/n : n \in \mathbb{N}\} = 1$.

   Solution: Since $1 > 1 - 1/n$ for each $n \in \mathbb{N}$, 1 is an upper bound of the set. Moreover, for each $\epsilon > 0$, by Archimedean Property, there exists $N \in \mathbb{N}$ s.t. $N \geq 1/\epsilon$. Then, $1 - 1/N > 1 - \epsilon$ and therefore $1 - \epsilon$ is not an upper bound of the set for each $\epsilon > 0$. We conclude that 1 is the supremum of the set $\{1 - 1/n : n \in \mathbb{N}\}$.

8. Let X be a nonempty set, and let f and g be defined on X and have bounded ranges in $\mathbb{R}$. Show that

$$\sup\{f(x) + g(x) : x \in X\} \leq \sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\}$$

   and that

$$\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq \inf\{f(x) + g(x) : x \in X\}.$$

   Give examples to show that each of these inequalities can be either equalities or strict inequalities.

   Solution: Since both f and g are bounded, their infimum and supremum exists in $\mathbb{R}$. Let s_f, s_g, i_f, $i_g \in \mathbb{R}$ be $\sup\{f(x) : x \in X\}$, $\sup\{g(x) : x \in X\}$, $\inf\{f(x) : x \in X\}$ and $\inf\{g(x) : x \in X\}$ respectively. By definition of infimum and supremum, we have $i_f \leq f(x) \leq s_f$ and $i_g \leq g(x) \leq s_g$ for all $x \in X$. Therefore, $i_f + i_g \leq f(x) + g(x) \leq s_f + i_f$ all $x \in X$. So $i_f + i_g \in \mathbb{R}$ is a lower bound of $\{f(x) + g(x) : x \in X\}$ and $s_f + s_g \in \mathbb{R}$ is an upper bound of $\{f(x) + g(x) : x \in X\}$. So we have the two desired inequalities.

   Let $f(x) = 0$ for every $x \in X$ be the zero function and g be any bounded function defined on X. Since $f(x) + g(x) = g(x)$ and $\sup\{f(x) : x \in X\} = \inf\{f(x) : x \in X\} = 0$, both inequalities are equalities.

   Let g be any bounded function defined on X s.t. $g(x_0) = 0$ for some $x_0 \in X$ and $g(x_1) = 1$ for some $x_1 \in X$. Let $f(x) = -g(x)$. Then $f(x) + g(x) = 0$ and $\sup\{f(x) + g(x) : x \in X\} = \inf\{f(x) + g(x) : x \in X\} = 0$. However, $f(x_0) = 0$ and $g(x_1) = 1$, $\sup\{f(x) : x \in X\} + \sup\{g(x) : x \in X\} \geq 0 + 1 > 0$. Also, $f(x_1) = -1$ and $g(x) = 0$, $\inf\{f(x) : x \in X\} + \inf\{g(x) : x \in X\} \leq 0 + (-1) < 0$.

9. Let $X = Y := \{x \in \mathbb{R} : 0 < x < 1\}$. Define $h : X \times Y \to \mathbb{R}$ by $h(x, y) := 2x + y$.

   (a) For each $x \in X$, find $f(x) := \sup\{h(x, y) : y \in Y\}$; then find $\inf\{f(x) : x \in X\}$.

   (b) For each $y \in Y$, find $g(y) := \inf\{h(x, y) : x \in X\}$; then find $\sup\{g(y) : y \in Y\}$. Compare with the result found in part (a).

   Solution:

   (a) By Example 2.4.1(a) and exercise 4, $f(x) = \sup\{2x+y : y \in Y\} = 2x+\sup\{y : y \in Y\}$. So $f(x) = 2x+1$. Then $\inf\{2x+1 : x \in X\} = 1+2\inf\{x : x \in X\} = 1$.

   (b) Similarly, $g(y) = \inf\{2x + y : x \in X\} = 2\inf\{x : x \in X\} + y = y$. Then $\sup\{y : y \in Y\} = 1$. The result is the same as part (a).

10. Perform the computations in (a) and (b) of the preceding exercise for the function $h : X \times Y \to \mathbb{R}$ defined by

$$h(x, y) := \begin{cases} 0 & \text{if } x < y, \\ 1 & \text{if } x \geq y. \end{cases}$$

Solution:

(a) For any x, $h(x, x) = 1$. So $f(x) = 1$ and thus $\inf\{f(x) : x \in X\} = 1$.

(b) For any $y \in Y$, $h(y/2, y) = 0$. So $g(y) = 0$ and thus $\sup\{g(y) : y \in Y\} = 0$. The result is smaller than that in part (a).

11. Let X and Y be nonempty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be defined by

$$f(x) := \sup\{h(x, y), y \in Y\}, \quad g(y) := \inf\{h(x, y) : x \in X\}.$$

Prove that

$$\sup\{g(y) : y \in Y\} \leq \inf\{f(x) : x \in X\}.$$

We sometimes express this by writing

$$\sup_y \inf_x h(x, y) \leq \inf_x \sup_y h(x, y).$$

Note that Exercises 9 and 10 show that the inequality may be either an equality or a strict inequality.

Solution: From example 2.4.1(b), it suffices to prove that $g(y) < f(x)$ for all $x \in X$ and $y \in Y$. Let $x_0 \in X$, $y_0 \in Y$ be given, we have $\inf_x h(x, y_0) \leq h(x, y_0)$ for every $x \in X$ and $\sup_y h(x_0, y) \geq h(x_0, y)$ for every $y \in Y$. In particular, $g(y_0) = \inf_x h(x, y_0) \leq h(x_0, y_0) \leq \sup_y h(x_0, y) = f(x_0)$.