MATH 2050A: Mathematical Analysis I (2017 1st term)

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1 Compact Sets in $\mathbb{R}$

Throughout this section, let (x_n) be a sequence in $\mathbb{R}$. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

Theorem 1.1 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

(i) : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) : $\lim_n (b_n - a_n) = 0$.

Then there is a unique real number ξ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. □

Theorem 1.2 (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof: See [1, Theorem 3.4.8]. □

Definition 1.3 A subset A of $\mathbb{R}$ is said to be *compact* (more precise, *sequentially compact*) if every sequence in A has a convergent subsequence with the limit in A.

We are now going to characterize the compact subsets of $\mathbb{R}$. The following is an important notation in mathematics.

Definition 1.4 A subset A is said to be *closed* in $\mathbb{R}$ if it satisfies the condition:

if (x_n) is a sequence in A and $\lim x_n$ *exists, then* $\lim x_n \in A$.

Example 1.5 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}$; $\mathbb{N}$; the empty set $\emptyset$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$.

(ii) (a, b) and $\mathbb{Q}$ are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

Proposition 1.6 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

The following is an important characterization of a compact set in $\mathbb{R}$. **Warning:** this result is not true for the so-called *metric spaces* in general.

Theorem 1.7 *Let A be a closed subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is compact.

(ii) A is closed and bounded.

Proof: It is clear that the result follows if $A = \emptyset$. So, we assume that A is non-empty.
For showing $(i) \Rightarrow (ii)$, assume that A is compact.
We first claim that A is closed. Let (x_n) be a sequence in A. Then by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. So, if (x_n) is convergent, then $\lim_n x_n = \lim_k x_{n_k} \in A$. Therefore, A is closed.
Next, we are going to show the boundedness of A. Suppose that A is not bounded. Fix an element $x_1 \in A$. Since A is not bounded, we can find an element $x_2 \in A$ such that $|x_2 - x_1| > 1$. Similarly, there is an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can obtain a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $m \neq n$. From this, we see that the sequence (x_n) does not have a convergent subsequence. In fact, if (x_n) has a convergent subsequence (x_{n_k}). Put $L := \lim_k x_{n_k}$. Then we can find a pair of sufficient large positive integers p and q with $p \neq q$ such that $|x_{n_p} - L| < 1/2$ and $|x_{n_q} - L| < 1/2$. This implies that $|x_{n_p} - x_{n_q}| < 1$. It leads to a contradiction because $|x_{n_p} - x_{n_q}| > 1$ by the choice of the sequence (x_n). Thus, A is bounded.
It remains to show $(ii) \Rightarrow (i)$. Suppose that A is closed and bounded.
Let (x_n) be a sequence in A. Thus, (x_n). Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence (x_{n_k}). Then by the closeness of A, $\lim_k x_{n_k} \in A$. Thus A is compact.
The proof is finished.
□

2 Appendix: Compact sets in $\mathbb{R}$, Part 2

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of $\mathbb{R}$, where Λ is an arbitrary non-empty index set, if each J_α is an open interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha.$$

Theorem 2.1 Heine-Borel Theorem: *Any closed and bounded interval $[a, b]$ satisfies the following condition:*

(HB) Given any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a, b]$, we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $[a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$

Proof: Suppose that $[a, b]$ does not satisfy the above Condition (HB). Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a, b]$ but it it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \le \xi \le b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished.
□

Remark 2.2 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.
For example, notice that $\{J_n := (1/n, 1) : n = 1, 2...\}$ is an open interval covers of $(0, 1)$ but you cannot find finitely many J_n's to cover the open interval $(0, 1)$.

The following is a very important feature of a compact set.

Theorem 2.3 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) For any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of A, we can find finitely many $J_{\alpha_1}, .., J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$.

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequent which has the limit in A. Put $X = \{x_n : n = 1, 2, ...\}$. Then X is infinite. Also, for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a - \delta_a, a + \delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a - \delta, a + \delta) \cap A$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a. On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the compactness of A, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. So we have $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, X must be finite. This leads to a contradiction. Therefore, A must be compact.
The implication $(ii) \Rightarrow (iii)$ follows from Theorem 1.7 at once.
It remains to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of A. Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Proposition 1.6. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a,b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 2.1, we can find finitely many J_α's and I_x's, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence A is compact.
The proof is finished. □

Remark 2.4 In fact, the condition in Theorem 2.3(i) is the usual definition of a *compact set* for a general topological space. More precise, if a set A satisfies the Definition 1.4, then A is said to be *sequentially compact.* Theorem 2.3 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of $\mathbb{R}$. However, these two notation are different for a general topological space.

Strongly recommended: take the courses: MATH 3060; MATH3070 for the next step.

3 Continuous functions defined on compact sets

Throughout this section, let A be a non-empty subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a function defined on A.

Proposition 3.1 *Let f be a continuous function defined on a compact subset A of $\mathbb{R}$. Then $f(A)$ is a compact subset of $\mathbb{R}$.*

Proof: **Method I**: By using Theorem 2.3 $(i) \Leftrightarrow (iii)$, it suffices to show that $f(A)$ is a closed bounded subset of $\mathbb{R}$.
Claim 1: $f(A)$ is bounded.
Suppose not. Then for each positive integer n, there is an element $x_n \in A$ such that $|f(x_n)| > n$.

Since A is compact, there is a convergent subsequence (x_{n_k}) with $a := \lim_k x_{n_k} \in A$. This gives $\lim_k f(x_{n_k}) = f(a)$ because f is continuous on a and hence, $(f(x_{n_k}))$ is a bounded sequence. This leads to a contradiction to the choice of (x_n) which satisfies $|f(x_{n_k})| > n_k$ for all $k = 1, 2....$
Claim 2: $f(A)$ is a closed subset of $\mathbb{R}$, that is, $y \in f(A)$ whenever, a sequence (x_n) in A satisfying $\lim_n f(x_n) = y$.
In fact, there is a convergent subsequence (x_k) with $z := \lim_k x_k \in A$ by using the compactness of A again. This gives $y = \lim_k f(x_{n_k}) = f(z) \in f(A)$ as desired since f is continuous on A.
Method II: Alternatively, we are going to use Theorem 2.3 $(i) \Leftrightarrow (ii)$.
Let $\{J_i\}_{i\in I}$ be an open interval covers of $f(A)$. We may assume $J_i \cap f(A) \neq \emptyset$ for each $i \in I$. Notice that since J_i is an open interval and f is continuous, we see that if $f(x) \in J_i$, then we can find $\delta_x > 0$ such that $f(z) \in J_i$ whenever $z \in A$ with $|z - x| < \delta_x$. Notice that we have $A \subseteq \bigcup_{x\in A} J_x$, where $V_x := (x - \delta_x, x + \delta_x)$ and hence, $\{V_x : x \in A\}$ forms an open intervals cover of A. By using the equivalence $(i) \Leftrightarrow (ii)$ in Theorem 2.3, we can find finitely many $x_1, ..., x_n$ in A such that $A \subseteq V_{x_1} \cup \cdots \cup V_{x_n}$. For each $k = 1, .., n$, then $f(x_k) \in J_{i_k}$ for some $i_k \in I$. Now if $x \in A$, then $x \in V_{x_k}$ for some $k = 1, ..., n$. This gives $f(x) \in J_{i_k}$ and thus, $f(A) \subseteq J_{i_1} \cup \cdots \cup J_{i_n}$. The proof is finished. □

Corollary 3.2 *If $f : A \to \mathbb{R}$ is a continuous injection and A is compact, then the inverse map $f^{-1} : f(A) \to A$ is also continuous.*

Proof: Let $B = f(A)$ and $g = f^{-1} : B \to A$. Suppose that g is not continuous at some $b \in B$. Put $a = g(b) \in A$. Then there are $\eta > 0$ and a sequence (y_n) in B such that $\lim y_n = b$ but $|g(y_n) - g(b)| \geq \eta$ for all n. Let $x_n := g(y_n) \in A$. So, by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} \in A$. Let $a' = \lim_k x_{n_k}$. Then we have $f(a') = \lim_k f(x_{n_k}) = \lim_k y_{n_k} = b$. On the other hand, since $|g(y_n) - g(b)| \geq \eta$ for all n, we see that

$$|x_{n_k} - a| = |g(y_{n_k}) - g(b)| \geq \eta > 0$$

for all k and hence $|a' - a| > 0$. This implies that $a \neq a'$ but $f(a') = b = f(a)$. It contradicts to f being injective.
The proof is finished. □

Remark 3.3 The assumption of the compactness in the last assertion of Proposition 3.2 is essential. For example, consider $A = [0, 1) \cup [2, 3]$ and define $f : A \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } \ x \in [0, 1) \\ x - 1 & \text{if } \ x \in [2, 3]. \end{cases}$$

Then $f(A) = [0, 2]$ and f is a continuous bijection from A onto $[0, 2]$ but $f^{-1} : [0, 2] \to A$ is not continuous at $y = 1$.

Example 3.4 By Proposition 3.2, it is impossible to find a continuous surjection from $[0, 1]$ onto $(0, 1)$ since $[0, 1]$ is compact but $(0, 1)$ is not. Thus $[0, 1]$ is not homeomorphic to $(0, 1)$.

Proposition 3.5 *Suppose that f is continuous on A. If A is compact, then there are points c and b in A such that*

$$f(c) = \max\{f(x) : x \in A\} \ \textit{and} \ f(b) = \min\{f(x) : x \in A\}.$$

Proof: By considering the function $-f$ on A, it needs to show that $f(c) = \max\{f(x) : x \in A\}$ for some $c \in A$.

Method I:

We first claim that f is bounded on A, that is, there is $M > 0$ such that $|f(x)| \le M$ for all $x \in A$. Suppose not. Then for each $n \in \mathbb{N}$, we can find $a_n \in A$ such that $|f(a_n)| > n$. Recall that A is compact if and only if it is closed and bounded (see Theorem 3.22). So, (a_n) is a bounded sequence in A. Then by the Bolzano-Weierstrass Theorem, there is a convergent subsequence (a_{n_k}) of (a_n). Put $a = \lim_k a_{n_k}$. Since A is closed and f is continuous, $a \in A$, from this, it follows that $f(a) = \lim_k f(a_{n_k})$. It is absurd because $n_k < |f(a_{n_k})| \to |f(a)|$ for all k and $n_k \to \infty$. So f must be bounded. So $L := \sup\{f(x) : x \in A\}$ must exist by the Axiom of Completeness.

It remains to show that there is a point $c \in A$ such that $f(c) = L$. In fact, by the definition of supremum, there is a sequence (x_n) in A such that $\lim_n f(x_n) = L$. Then by the Bolzano-Weierstrass Theorem again, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. If we put $c := \lim_k x_{n_k} \in A$, then $f(c) = \lim_k f(x_{n_k}) = L$ as desired. The proof is finished.

Method II:

We first claim that f is bounded above. Notice that for each $x \in A$, there is $\delta_x > 0$ such that $f(y) < f(x) + 1$ whenever $y \in A$ with $|x - y| < \delta_x$ since f is continuous on A. Now if we put $J_x := (x-\delta_x, x+\delta_x)$ for each $x \in A$, then $A \subseteq \bigcup_{x\in A} J_x$. So, by the compactness of A, we can find finitely many $x_1, ..., x_N$ in A such that $A \subseteq J_{x_1} \cup \cdots \cup J_{x_N}$ and it follows that for each $x \in A$, we have $f(x) < 1+f(x_k)$ for some $k = 1, ..., N$. Now if we put $M := \max\{1+f(x_1), ..., 1+f(x_N)\}$, then f is bounded above by M on A.

Put $L := \sup\{f(x) : x \in A\}$. It remains to show that there is an element $c \in A$ such that $f(c) = L$. Suppose not. Notice that since $f(x) \le L$ for all $x \in A$, we have $f(x) < L$ for all $x \in A$ under this assumption. Therefore, by the continuity of f, for each $x \in A$, there are $\varepsilon_x > 0$ and $\eta_x > 0$ such that $f(y) < f(x) + \varepsilon_x < L$ whenever $y \in A$ with $|y - x| < \delta_x$. Put $I_x := (x-\eta_x, x+\eta_x)$. Then $A \subseteq \bigcup_{x\in A} I_x$. By the compactness of A again, A can be covered by finitely many $I_{x_1}, ..., I_{x_N}$. If we let $L' := \max\{f(x_1)+\varepsilon_{x_1}, ..., f(x_N)+\varepsilon_{x_N}\}$, then $f(x) < L' < L$ for all $x \in A$. It contradicts to L being the least upper bound for the set $\{f(x) : x \in A\}$. The proof is complete. □

Definition 3.6 We say that a function f is *upper semi-continuous (resp. lower semi-continuous)* on A if for each element $z \in A$ and for any $\varepsilon > 0$, there is $\delta > 0$ such that $f(x) < f(z) + \varepsilon$ (resp. $f(z) - \varepsilon < f(x)$) whenever $x \in A$ with $|x - z| < \delta$.

Remark 3.7

(i) It is clear that a function is continuous if and only if it is upper semi-continuous and lower semi-continuous. However, an upper semi-continuous function need not be continuous. For example, define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

(ii) From the **Method II** above, we see that if f is upper semi-continuous (resp. lower semi-continuous) on a compact set A, then the function f attains the supremum (resp. infimum) on A.

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4*th ed*), Wiley, (2011).