Math 4010: Functional Analysis: Test 05 Nov 2018. 10:30-12:00

Answer ALL Questions

Throughout the paper, let B_Z be the closed unit ball of a normed space Z and let $B(a,r) := \{x \in Z : \|x-a\| < r\}$ for $a \in Z$ and $r > 0$.

1. (10 points) Let X ba a normed space. Show that the following statements are equivalent.

   (i) X is a Banach space.

   (ii) Every absolutely convergent series in X is convergent, that is, if (x_n) is a sequence in X such that $\sum_{k=1}^{\infty} \|x_k\| < \infty$, then the limit $\lim\limits_{n\to\infty} \sum_{k=1}^{n} x_k$ exists in X.

   (Hint: Using a known fact that a Cauchy sequence is convergent if and only if it has a convergent subsequence.)

   Proof: See lecture note: Prop 1.1 □

2. (i) (5 points): Let E be a normed space. Let $0 < r < 1$. Suppose that $B_{E^*} \subseteq \bigcup_{k=1}^{N} B(x_k^*, r)$ for some finitely many elements $x_1^*, ..., x_N^*$ in B_E^*. Define a map $T : E \to c_0$ by

$$T(x) := (x_1^*(x), ..., x_N^*(x), 0, 0, ...) \in c_0$$

   for $x \in E$. Show that $(1-r)\|x\| \le \|Tx\|_\infty \le \|x\|$ for all $x \in E$.

   (ii) (5 points): Let X be a normed space. Show that for any finite dimensional subspace E of X and for any $\eta > 0$, there exist a finite dimensional subspace F of c_0 and an isomorphism T from E onto F so that $\|T\|\|T^{-1}\| < 1+\eta$.

   Proof: (i): Let $x \in E$. Then $|x_k^*(x)| \le \|x_k^*\|\|x\| \le \|x\|$ for all $k = 1, 2, ...$ and hence, we see that $\|Tx\|_\infty \le \|x\|$.

   On the other hand, Hahn-Banach separation tells us that $\|x\| = |f(x)|$ for some $f \in X^*$ with $\|f\| = 1$. By the assumption, we have $\|f - x_k^*\| < r$ for some $k = 1, ..., N$. This gives

$$\|x\| = |f(x)| \le |(f - x_k^*)(x)| + |x_k^*(x)| \le r\|x\| + \|Tx\|_\infty.$$

   So, we have $(1-r)\|x\| \le \|Tx\|_\infty$ as required.

   (ii): Let $0 < r < 1$. If E is of finite dimensional, then so is E^*. Then by the compactness of B_{E^*}, there exist finitely many elements $x_1^*, ..., x_N^*$ in B_{E^*} such that $B_{E^*} \subseteq \bigcup_{k=1}^{N} B(x_k^*, r)$. Let T be defined as in (i), we see that $\|T\| \le 1$. Moreover, if we let $F := T(E)$, then T is an isomorphism from E onto F with $\|T^{-1}\| \le \frac{1}{1-r}$. Notice that $\lim_{r\to 0+} \frac{1}{1-r} = 1+$. Hence, for any $\eta > 0$, choose $r > 0$ such that $1 < \frac{1}{1-r} < 1+\eta$ and thus, $\|T^{-1}\| < 1+\eta$ as desired. The proof is finished. □

3. Let $1 < p < \infty$. For each $x \in c_0$, define a linear operator M_x from ℓ^p to itself by

$$M_x(\xi)(k) := x(k)\xi(k)$$

   for $\xi \in \ell^p$ and $k = 1, 2.....$

(i) (5 points) Show that $\|M_x\| = \|x\|_\infty$ for any $x \in c_0$.

(ii) (5 points) What is $M_x^*\xi^*$ for $x \in c_0$ and $\xi^* \in (\ell^p)^*$?

Proof: (*i*) Let $x \in c_0$. It is clear that $\|M_x\| \leq \|x\|_\infty$ because we always have $\|M_x(\xi)\|_p^p = \sum_{k=1}^{\infty} |x(k)\xi(k)|^p \leq \|x\|_\infty \|\xi\|_p^p$ for all $\xi \in \ell^p$. On the other hand, given any $\varepsilon > 0$, we have $\|x\|_\infty - \varepsilon < |x(N)|$ for some positive integer N. Let (e_k) be the canonical Schauder basis for ℓ^p. Then we see that $\|x\|_\infty - \varepsilon < |x(N)| = M_x(e_N) \leq \|M_x\|$ for all $\varepsilon > 0$. This implies that $\|x\|_\infty \leq \|M_x\|$. Part (*i*) follows.
(*ii*) Let $J : (\ell^p)^* \to \ell^q$ be the canonical isometric isomorphism, where $1/p + 1/q = 1$. Now if we put $m_x(\eta)(k) := x(k)\eta(k)$ for $\eta \in \ell^q$ and $k = 1, 2....$ Then we have the following commutative diagram:

$$\begin{array}{ccc} (\ell^p)^* & \xrightarrow{J} & \ell^q \\ \downarrow M_x^* & & \downarrow m_x \\ (\ell^p)^* & \xrightarrow{J} & \ell^q \end{array}$$

So, $M_x^* = m_x$ under the identification J. □

End