Solution 2

p.105: 1, 5

1. Show that the following sets are closed subspaces of their respective spaces:

   (a) $\{ (a_i) \in \ell^\infty : a_0 = 0 \}$,

   (b) $\{ (a_i) \in \ell^2 : a_1 = a_3 \text{ AND } a_0 = \sum_{i=1}^\infty a_i/i \}$,

   (c) $\{ f \in C[0,1] : \int_0^1 f = 0 \}$.

   Solution.

   (a) Let $A = \{ (a_i) \in \ell^\infty : a_0 = 0 \}$.
   First, we check that A is a subspace of ℓ^∞. Let $\lambda \in \mathbb{K}$ and $x, y \in A$. We are going to show that

   (i) $x + y \in A$,

   (ii) $\lambda x \in A$.

   We may clarify the notation x and y here. Note that x, y are elements of A, and hence elements of ℓ^∞. That means x, y are sequences of numbers. We denote by $x(i)$ and $y(i)$ the $(i+1)$th entry of the sequences x and y respectively.

   Now, A is the set of ℓ^∞-sequences such that its first entry is 0.

   The first entry of the sequence $x+y$ is by definition $(x+y)(0) = x(0)+y(0) = 0$. This shows (i).

   The first entry of λx is by definiton $\lambda x(0) = 0$. This shows (ii).

   To see that A is closed, suppose (x_n) is a sequence in A and (x_n) converges to some $x \in \ell^\infty$. It suffices to show that $x \in A$. Note that for every $n \in \mathbb{N}$,

   $$|x(0)| = |x(0) - x_n(0)| \leq \sup_{0 \leq k < \infty} |x(k) - x_n(k)| = \|x - x_n\|_{\ell^\infty}.$$

   As $n \to \infty$, $\|x - x_n\|_{\ell^\infty} \to 0$ and hence we can conclude that $|x(0)| \leq 0$. This shows $x(0) = 0$ and hence $x \in A$.

   (b) Let $B = \{ (a_i) \in \ell^2 : a_1 = a_3 \text{ AND } a_0 = \sum_{i=1}^\infty a_i/i \}$.
   We only show that B is closed. Suppose (x_n) is a sequence in B and (x_n) converges to some $x \in \ell^2$. It suffices to show that $x \in B$.

   First, we claim that $x(i) = \lim_{n\to\infty} x_n(i)$ for each fixed i. Note that

   $$|x(i) - x_n(i)| \leq \sqrt{\sum_{k=0}^\infty |x(k) - x_n(k)|^2} = \|x - x_n\|_{\ell^2}.$$

   By the convergence of x_n to x in ℓ^2, we see that $x(i) = \lim_{n\to\infty} x_n(i)$.

In particular, this gives us that $x(1) = \lim_{n\to\infty} x_n(1) = \lim_{n\to\infty} x_n(3) = x(3)$. The second equality holds because every x_n is an element in B.

It remains to show that $x(0) = \sum_{i=1}^{\infty} x(i)/i$. To do so, let $\epsilon > 0$, we would like to check that for any sufficiently large $N \in \mathbb{N}$, we have

$$\left| x(0) - \sum_{i=1}^{N} \frac{x(i)}{i} \right| < \epsilon.$$

For any $n, N \in \mathbb{N}$, we have

$$\left| x(0) - \sum_{i=1}^{N} \frac{x(i)}{i} \right| \leq |x(0) - x_n(0)| + \left| x_n(0) - \sum_{i=1}^{N} \frac{x_n(i)}{i} \right| + \left| \sum_{i=1}^{N} \frac{x_n(i)}{i} - \sum_{i=1}^{N} \frac{x(i)}{i} \right|$$

Let

$$\begin{aligned} \text{(I)} &= |x(0) - x_n(0)|, \\ \text{(II)} &= \left| x_n(0) - \sum_{i=1}^{N} \frac{x_n(i)}{i} \right|, \\ \text{(III)} &= \left| \sum_{i=1}^{N} \frac{x_n(i)}{i} - \sum_{i=1}^{N} \frac{x(i)}{i} \right|. \end{aligned}$$

We now estimate the bound for (I), (II), (III). For (I), we can fix a large $n \in \mathbb{N}$ such that (I) $< \epsilon/3$. For (II), when n is fixed and N goes to infinity, (II) will go to 0, due to $x_n(0) = \sum_{i=1}^{\infty} \frac{x_n(i)}{i}$. For (III), and for each fixed N, (III) is small when n is large, but it is not what we want. We should fix an $n \in \mathbb{N}$ and let N go to infinity. Thus, we need a better estimation for (III).

$$\begin{aligned} \left| \sum_{i=1}^{N} \frac{x_n(i)}{i} - \sum_{i=1}^{N} \frac{x(i)}{i} \right| &= \left| \sum_{i=1}^{N} \frac{1}{i} \left(x_n(i) - x(i) \right) \right| \\ &\leq \sqrt{\sum_{i=1}^{N} \left(\frac{1}{i} \right)^2} \sqrt{\sum_{i=1}^{N} |x(i) - x_n(i)|^2} \\ &\leq C \|x - x_n\|_{\ell^2}, \text{ where } C := \sqrt{\sum_{i=1}^{\infty} \left(\frac{1}{i} \right)^2} \end{aligned}$$

By the calculation, (III) can be well-controlled when n is sufficiently large and this bound is independent of N. To conclude, for an $\epsilon > 0$, we can fix a sufficiently large n such that both (I), (III) $< \epsilon/3$, independent of N. For this fixed n, there is some $N_0 \in \mathbb{N}$ such that (II) $< \epsilon/3$ when $N \geq N_0$. Therefore,

$$\left| x(0) - \sum_{i=1}^{N} \frac{x(i)}{i} \right| \leq (I) + (II) + (III) < \epsilon, \quad \text{when } N \geq N_0.$$

Finally, we check that $\sum_{i=1}^{\infty} \left(\frac{1}{i}\right)^2 < \infty$. Note that for every $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{i=1}^{N} \left(\frac{1}{i}\right)^2 &= 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \ldots + \left(\frac{1}{N}\right)^2 \\ &\leq 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(N-1)N} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\ &= 2 - \frac{1}{N} \leq 2\end{aligned}$$

In the last equality, the sum telescopes.

(c) Let $C = \{ f \in C[0,1] : \int_0^1 f = 0 \}$.
We only show that C is closed in $C[0,1]$. Suppose that (f_n) is a sequence in C and converges to some $f \in C[0,1]$. It suffices to show that $f \in C$.
Recall that the norm on $C[0,1]$ is the supnorm,
i.e. $\|g\|_{C[0,1]} = \sup_{x \in [0,1]} |g(x)|$ for $g \in C[0,1]$.
Since $f_n \in C$, we have $\int_0^1 f_n = 0$ for all n. Note then

$$\begin{aligned}\left|\int_0^1 f\right| &= \left|\int_0^1 f - \int_0^1 f_n\right| \\ &\leq \int_0^1 |f - f_n| \\ &\leq \int_0^1 \|f - f_n\|_{C[0,1]} \\ &= \|f - f_n\|_{C[0,1]}\end{aligned}$$

Since $\lim_{n\to\infty} \|f - f_n\|_{C[0,1]} = 0$, we have $\int_0^1 f = 0$ and therefore $f \in C$.

5. The continuity of $+$ and $\lambda\cdot$ imply that $\overline{\lambda A} = \lambda \overline{A}$ and $\overline{A} + \overline{B} \subseteq \overline{A + B}$. Find an example to show that equality need not necessarily hold.

Solution.

Let A, B be nonempty sets.

If $\lambda = 0$, then $\overline{\lambda A} = \overline{\{0\}} = \{0\}$ and $\lambda \overline{A} = \{0\}$, i.e. $\overline{\lambda A} = \lambda \overline{A}$.

For $\lambda \neq 0$, we first show that $\lambda \overline{A} \subseteq \overline{\lambda A}$.

The continuity of scalar multiplication, according to Proposition 7.8 of our textbook, means that if (λ_n) and (x_n) are sequences of scalars and vectors with λ_n converging to λ and x_n converging to x, then $\lambda_n x_n$ converges to λx.

Pick any element in $\lambda \overline{A}$. It can be written as λx for some $x \in \overline{A}$. That is, we can find a sequence (x_n) in A such that x_n converges to x. By the continuity of scalar multiplication, we see that

$$\lim_{n\to\infty} (\lambda x_n) = \left(\lim_{n\to\infty} \lambda\right)\left(\lim_{n\to\infty} x_n\right) = \lambda x.$$

This shows that λx is the limit of the sequence (λx_n) in λA. Therefore, $\lambda x \in \overline{\lambda A}$. The above gives us $\lambda \overline{A} \subseteq \overline{\lambda A}$.

The other way round, let $y \in \overline{\lambda A}$. Then, y is the limit of a sequence (λx_n) with $x_n \in A$. Since $\lambda \neq 0$, by the continuity of scalar multiplication,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{\lambda}(\lambda x_n) = \frac{1}{\lambda} \lim_{n\to\infty} \lambda x_n = \frac{1}{\lambda} y.$$

This shows that $\frac{1}{\lambda} y \in \overline{A}$ and therefore, $y = \lambda(\frac{1}{\lambda} y) \in \lambda \overline{A}$.

We have obtained $\overline{\lambda A} = \lambda \overline{A}$ for nonzero λ.

Continuity of vector addition tells us that if (x_n) and (y_n) are sequences of vectors with limit x and y respectively, then $x_n + y_n$ converges to $x + y$.

To see that $\overline{A} + \overline{B} \subseteq \overline{A + B}$, let $a \in \overline{A}$ and $b \in \overline{B}$. There exists two sequences (a_n) in A and (b_n) in B with limits a and b respectively. By the continuity of vector addition,

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b.$$

This shows that $a+b$ is the limit of a sequence in $A+B$. That sequence is (a_n+b_n). Therefore, $a + b \in \overline{A + B}$.

For an example of A, B with $\overline{A} + \overline{B} \subsetneq \overline{A + B}$. Let

$$\begin{aligned} A &= \{n - 1/n \in \mathbb{R} : n = 2, 3, \ldots\} \\ B &= \{n \in \mathbb{R} : n = -1, -2, \ldots\} \end{aligned}$$

We now argue that A is a closed set. Suppose z_0 is a limit point of A. Then, there are infinitely many points $z \in A$ such that $0 < |z - z_0| < 1/2$. Let z_1, z_2 be two such points and $z_1 \neq z_2$, then by triangle inequality,

$$|z_1 - z_2| \leq |z_1 - z_0| + |z_2 - z_0| < \frac{1}{2} + \frac{1}{2} = 1.$$

Note then if $z_1, z_2 \in A$ with $|z_1 - z_2| < 1$, then $z_1 = z_2$. This contradicts to our assumption $z_1 \neq z_2$. Therefore, A has no limit points and A itself is a closed set. Similarly, one can show that B is closed.

Finally, we argue that $0 \in \overline{A + B}$ but $0 \notin \overline{A} + \overline{B}$.

For $0 \in \overline{A + B}$, Let $(x_n) = (n + 1 - 1/(n + 1))$, $(y_n) = (-n - 1)$ be sequences in A and B respectively. $(x_n + y_n) = (-1/(n + 1))$ is a sequence in $A + B$ with limit equal to 0.

For $0 \notin \overline{A} + \overline{B}$, we have argued that A, B are closed sets, i.e. $\overline{A} = A$ and $\overline{B} = B$. Suppose $0 \in \overline{A} + \overline{B}$. We have

$$n + 1 - \frac{1}{n + 1} - m = 0 \quad \text{for some } n, m \in \mathbb{N}.$$

That is, $n + 1 - m = 1/(n + 1)$. From RHS, we see that $n + 1 - m$ is a number in $(0, 1/2]$, but $n + 1 - m$ is an integer. This shows that $0 \notin \overline{A} + \overline{B}$.