1 Basic Topological Notions and Examples

In this note we use $B_r(x) := (x-r, x+r)$ for the open neighborhood centered at $x \in \mathbb{R}$ with radius $r > 0$.

Definition 1.1 (Basic Definitions)**.** Let $A \subset \mathbb{R}$. Then

- we call A a *closed set* if for all sequence (x_n) in A such that $\lim x_n = x \in \mathbb{R}$, then $x \in A$.
- we call $\bar{A}$ the *closure* of A if $\bar{A}$ is the largest closed set. Equivalently, $\bar{A} = A \cup D(A)$ where $D(A)$ is called the derived set of A, which contains all sequential limits of A.
- We call A a *dense subset* if $\bar{A} = \mathbb{R}$. Equivalently, for all open interval $I \subset \mathbb{R}$, we have that $I \cap A \neq \phi$.
- we call A an *open set* if for all $x \in A$ there exists $r > 0$ such that $x \in B_r(x) \subset A$.
- we call $a \in \mathbb{R}$ a *cluster point* if for all open interval $I \ni a$ we have $I \cap A \backslash \{a\} \neq \phi$. Equivalently, there exists (x_n) in $A \backslash \{a\}$ such that $\lim x_n = a$.
- we call $a \in A$ an *isolated point* if a is not a cluster point. Equivalently, there exists $r > 0$ such that $B_r(x) \cap A = \{a\}$.

Proposition 1.2. *We have the following basic set-theoretic results for these basic topological notions:*

i. A subset A is closed if and only if its complement A^c is open. Therefore $\mathbb{R}$ and ϕ are open and closed (or clopen) sets.

ii. The union of two closed sets (and hence finitely many closed sets) is closed while the intersection of two open sets (and hence finitely many open sets) is open.

iii. The intersection of any (not necessarily finite) collection of closed sets is closed while the union of any collection of open sets is open.

Quick Practice. *Verify all equivalent formulations stated in Definition 1.1 and prove Proposition 1.2.*

Definition 1.3 (Continuous Functions)**.** Let $f : A \to \mathbb{R}$ be a function defined on $A \subset \mathbb{R}$.

- We say that f is *continuous at* $x \in A$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in A$ with $|x - y| < \delta$. In other words, $f(y) \in B_\epsilon(f(x))$ for all $y \in B_\delta(x) \cap A$.
- We say that f is *continuous on* A if f is continuous for all $a \in A$.

Example 1.4. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function. Show that f is continuous.*

Solution. We have to show that f is continuous at every point $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$. Let $\epsilon > 0$. Then take $\delta := 1$. It follows that $B_\delta(n) \cap \mathbb{N} = \{n\}$ (why?). Therefore for all $y \in B_\delta(n) \cap \mathbb{N}$, we have $y = n$ and so $|f(y) - f(n)| = |f(n) - f(n)| = 0 < \epsilon$.

Theorem 1.5 (Characterization of Continuity)**.** *Let $f : A \to \mathbb{R}$ be a function and $a \in A$.*

i. Then f is continuous at a if and only if for all sequences (x_n) in A that converges to a, we have $\lim f(x_n) = f(a)$.

ii. If a is a cluster point of A, then f is continuous if and only if $\lim_{x\to a} f(x) = f(a)$

Remark. We do not have to consider clustering of points when using the sequential criteria for continuity.

Quick Practice. *Let $f : A \to \mathbb{R}$ be a function with $a \in A$ isolated. Show that f is continuous at $a \in A$ using two different proofs, one with only the definition of continuity and one with the sequential criteria.*

Example 1.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$. Show that $f \equiv 0$.*

Solution. Suppose not. Then $f(a) > 0$ for some $a \notin \mathbb{Q}$. Note that a is a cluster point of $\mathbb{R}$. Hence, $\lim_{x\to a} f(x) > 0$. It follows that there exists $r > 0$ such that $f(x) > 0$ for all $x \in B_r(a)$ (why?). Howerver, $B_r(a)$ is an open interval and so by density of $\mathbb{Q}$, we have $B_r(a) \cap \mathbb{Q} \neq \phi$, which is not possible as f vanishes on $\mathbb{Q}$. It must be the case that $f \equiv 0$ on $\mathbb{R}$.

Quick Practice.

1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that f is a continuous function if and only if $f^{-1}(U)$ is an open set for all open sets $U \subset \mathbb{R}$.*
2. *Let $f, g : \mathbb{R} \to \mathbb{R}$. Suppose $f = g$ on some dense set $D \subset \mathbb{R}$. Show that $f = g$ on $\mathbb{R}$*

Example 1.7 (Dirichlet Function)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

In other words, $f = \chi_{\mathbb{Q}}$ is the characteristic function of $\mathbb{Q}$. Show that f is discontinuous everywhere.

Solution. Suppose $x \in \mathbb{Q}$. Then $f(x) = 1$. Since $\mathbb{R}\backslash\mathbb{Q}$ is dense, there exists (α_n) irrational such that $\lim \alpha_n = x$. However, $\lim f(\alpha_n) = \lim 0 = 0 \neq 1 = f(x)$, which violates the sequential criteria. The case for $x \notin \mathbb{Q}$ follows from the density of $\mathbb{Q}$ and is left as an exercise.

Example 1.8 (Thomae's Function)**.** *Let $f : [0, 1] \to \mathbb{R}$ be defined by*

$$f(x) := \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{n} & x \in \mathbb{Q}, x = \frac{m}{n} \text{ with } \gcd(m, n) = 1 \end{cases}$$

Find the points of continuity of f, that is, those points where f is continuous at.

Solution. The point of continuity if precisely $\mathbb{R}\backslash\mathbb{Q}$.
We first show that f is not continuous on $\mathbb{Q}$. Let $x \in [0, 1] \cap \mathbb{Q}$. Then $f(x) = 1/n$ for some $n \in \mathbb{N}$ and so $f(x) > 0$. The discontinuity can be shown by considering some irrational sequence converging to x using the sequential criteria(why?).
Next we move on to the hard part, which is the continuity at the irrationals. To this end we define

$$F_n := \{x \in [0, 1] : x = \frac{m}{n}, \gcd(m, n) = 1\}$$

for all $n \in \mathbb{N}$. Then it is not hard to see that F_n are finite sets for all $n \in \mathbb{N}$ with $[0, 1] \cap \mathbb{Q} = \bigcup_n F_n$ (why?). Now let $\alpha \in [0, 1]\backslash\mathbb{Q}$. Let $\epsilon > 0$. Then we pick $N \in \mathbb{N}$ such that $1/N < \epsilon$ by the Archimedean Property. Now write $B := \bigcup_{i=1}^{N} F_i$. Then B is a finite set. Then $\inf\{|x - b| : b \in B\} = \min\{|x - b| : b \in B\} > 0$ (why?). Note we take $0 < \delta < \min\{|x - b| : b \in B\}$. Suppose we have $y \in B_\delta(x)$. If $y \notin \mathbb{Q}$. Then $|f(y) - f(\alpha)| = 0 < \epsilon$. If $y \in \mathbb{Q}$, it must be the case that $y \in \bigcup_{i>N} F_i$ by the choice of δ (why?). Hence, $f(y) = \frac{1}{i}$ for some $i \geq N$. It follows that $|f(y) - f(\alpha)| = \frac{1}{i} \leq \frac{1}{N} < \epsilon$.

2 Exercise

1. Show that every finite set of $\mathbb{R}$ is closed. Find an example of a countably infinte subset that is closed.
2. Let $\phi \neq A \subset \mathbb{R}$. We define $d_A : \mathbb{R} \to \mathbb{R}$ by $d_A(x) := \inf\{|x - a| : a \in A\}$.
   (a) Show that d_A is well-defined, that is, $d_A(x) < \infty$ for all $x \in \mathbb{R}$.
   (b) Show that $d_A(x) = 0$ if and only if $x \in \overline{A}$.
   (c) Show that $|d_A(x) - d_A(y)| \leq |x - y|$ and deduce that d_A is a continuous function.
3. Let $\{F_n\}$ be an increasing sequence of closed set, that is, $F_n \subset F_m$ for all $n \leq m$. Let $A := \bigcup_n F_n$. Define $f : \mathbb{R} \to \mathbb{R}$ by

   $$f(x) = \begin{cases} \frac{1}{n} & x \in \mathbb{Q} \cap A \text{ and } n \text{ is the minimal natural number such that } x \in F_n \\ -\frac{1}{n} & x \in A\backslash\mathbb{Q} \text{ and } n \text{ is the minimal natural number such that } x \in F_n \\ 0 & x \notin A \end{cases}$$

   Show that the point of continuity of f is **precisely** $\mathbb{R}\backslash A$.
   (Hint: Question 2 can be useful and the function here is from the Thomae's Function entry in Wikipedia)
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.
   (a) Show that f is continuous at $a \in \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in B_\delta(a)$.
   (b) Define $U_\epsilon := \{a \in \mathbb{R} : \text{there exists } \delta > 0 \text{ such that } |f(x) - f(y)| < \epsilon \text{ for all } x, y \in B_\delta(a)\}$ for all $\epsilon > 0$. Show that U_ϵ are open sets for all $\epsilon > 0$ and deduce that the point of continuity of f is always the intersection of **countably many** open sets.
5. (Extremely Challenging) Let (U_n) be a sequence of subsets that are both open and dense.
   (a) Show that $\bigcap U_n$ is a dense set. *(Hint: You may want to use the nested interval theorem).*
   (b) Hence, show that $\mathbb{R}\backslash\mathbb{Q}$ cannot be the union of countably many closed sets. Deduce further that there is no real-valued functions with point of continuity being precisely the set of rational numbers.