# 1 Eventual and Frequent Behavior of Sequences

**Definition 1.1.** Let $(x_n)$ be a sequence. Let $P := \{x \in \mathbb{R} : x \text{ satisfies property (P)}\}$. Then

- We say $(x_n)$ satisfies property (P) **eventually**, or for sufficient large $n$, if there exists $N \in \mathbb{N}$ such that $x_n \in P$ for all $n \geq N$
- We say $(x_n)$ satisfies property (P) **frequently** if for all $N \in \mathbb{N}$ there exists $k(N) \geq N$ such that $x_{k(N)} \in P$

**Quick Practice.** *Let $(x_n)$ be a sequence. Let $P := \{x \in \mathbb{R} : x \text{ satisfies property (P)}\}$. Show that*

a. *$(x_n)$ satisfies property (P) eventually if and only if $(x_n)$ does not satisfy property (P) for only* ***finitely*** *many terms*

b. *$(x_n)$ satisfies property (P) frequently if and only if $(x_n)$ satisfies (P) for* ***infinitely*** *many terms*

c. *$(x_n)$ satisfies property (P) frequently if and only if there exists a* ***subsequence*** *$(y_n)$ of $(x_n)$ such that $(y_n)$ satisfies (P) for all terms*

d. *The negation of $(x_n)$ satisfies property (P) eventually is that $(x_n)$ does not satisfies (P) frequently.*

**Example 1.2.** *Let $(x_n)$ be a convergent sequence. Write $x := \lim x_n$. Suppose $x > r$. Show that $x_n > r$ for sufficiently large $n$.*

*Solution.* Recall that $\lim x_n = \liminf x_n$ as $(x_n)$ is convergent. Therefore, $\liminf x_n > r$. From Tutorial 3, this shows that $x_n \leq r$ for only finitely many terms (why?). This is equivalent to that $x_n > r$ eventually.
Alternatively, we can prove the assertion using an $\epsilon-$ argument. Note that $x > r$ and so $L := x - r > 0$. Therefore, there exists $N \in \mathbb{N}$ such that $|x_n - x| < L/2$. This implies $-L/2 + x < x_n < L/2 + x$. Note that $-L/2 + x = (r + x)/2 > r$. Therefore, we have $x_n > r$ for all $n \geq N$.

**Example 1.3.** *Let $(x_n)$ be a sequence such that $x_n \geq 0$ for all $n \in \mathbb{N}$. Define $y_n := x_n^{1/n}$ for all $n \in \mathbb{N}$. Suppose $\limsup y_n < 1$. Then show that $\lim x_n = 0$.*

*Proof.* Let $\limsup y_n < r < 1$. Then $\limsup y_n < r < 1$, it follows that $y_n < r$ for sufficiently large $n$ (why?). Therefore there exists $N \in \mathbb{N}$ such that $y_n < r$, which imples $x_n^{1/n} < r \iff x_n < r^n$ for all $n \geq N$. As a result we have $0 \leq x_n \leq r^n$ for all $n \geq N$. Since $r \in (0, 1)$ (why?), it follows that $\lim r_n = 0$. By Squeeze Theorem, we have $\lim x_n = 0$. □

**Definition 1.4.** Let $(x_n)$ be a sequence in $\mathbb{R}$. Let $x \in \mathbb{R}$. Then we say $x$ a (sequential) **cluster point** of $(x_n)$ if for all $\epsilon > 0$ we have $|x - x_n| < \epsilon$ frequently.

**Example 1.5.** *Let $(x_n)$ be a sequence and $x \in \mathbb{R}$ a cluster point of $(x_n)$. Show that there exists a subsequence $(y_n)$ such that $\lim x_n = x$.*

*Solution.* By definition of a cluster point, there exists $N(1) \in \mathbb{N}$ such that $\left|x - x_{N(1)}\right| < 1$. Note that $x$ is also a cluster point of the tail sequence $(x_n)_{n>N(1)}$. By definition again, there exists $N(2) \in \mathbb{N}$ such that $\left|x - x_{N(2)}\right| < 1/2$ where $N(2) > N(1)$. Again, $x$ is a cluster point of the tail sequence $(x_n)_{n>N(2)}$. Inductively for $n \geq 3$, there exists $N(n) \in \mathbb{N}$ such that $\left|x - x_{N(n)}\right| < 1/n$ with $N(n) > N(n-1)$.
Now we define $y_n := x_{N(n)}$. Then $(y_n)$ is a subsequence of $(x_n)$ as $n \mapsto N(n)$ is strictly increasing by construction. It remains to show that $(y_n)$ is our desired convergent subsequence. Note that we have $0 \leq |y_n - x| \leq 1/n$ for all $n \in \mathbb{N}$. By Squeeze Theorem, it follows that $\lim |y_n - x| = 0$ and so $\lim y_n = x$ (why?).

**Quick Practice.**

1. *Let $(x_n)$ be a convergent sequence. Show that if $\lim x_n < r$ for $r \in \mathbb{R}$, then $x_n < r$ eventually.*
2. *Let $(x_n)$ be a bounded sequence. Suppose $\overline{\lim}\, x_n < r$ for $r \in \mathbb{R}$. Show that $x_n < r$ eventually.*
3. *Let $(x_n)$ be a sequence such that $x_n > 0$ for all $n \in \mathbb{N}$. Suppose $\limsup \frac{x_{n+1}}{x_n} < 1$. Show that $\lim x_n = 0$*
4. *Let $(x_n)$ be a sequence and $x \in \mathbb{R}$. Show that $x$ is a sequential cluster point of $(x_n)$ if and only if it is the limit of some subsequence of $(x_n)$.*
5. *Let $\epsilon > 0$ and $(x_n)$ a sequence. Define $A_{n,\epsilon} := \{y \in \mathbb{R} : |x_n - y| < \epsilon\}$ and $A := \bigcap_\epsilon \bigcup_n \bigcap_{k \geq n} A_{k,\epsilon}$. What is $A$ and the complement of $A$?*

## 2 Miscellaneous Examples and Exercises on Subsequences

**Example 2.1.** *Let $(x_n)$ be a sequence in $\mathbb{R}$. Suppose every subsequence of $(x_n)$ has a further subsequence that converges to 0. Show that $(x_n)$ converges to 0.*

*Solution.* Suppose not. Then there exists $\epsilon > 0$ such that $|x_n| \geq \epsilon$ frequently. In other words, there exists a subsequence $(y_n)$ of $(x_n)$ such that $|y_n| \geq \epsilon$ for all $n \in \mathbb{N}$. By assumption, there exists a further subsequence $(z_n)$ of $(y_n)$ such that $\lim z_n = 0$. However as $(z_n)$ is a subsequence, we have $|z_n| \geq \epsilon > 0$ for all $n \in \mathbb{N}$. By order property of limits $\lim z_n = 0 \geq \epsilon > 0$, which is a contradiction.

**Example 2.2** (Showing the Nested Interval Theorem by the B-W Theorem)**.** *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded interval such that $I_n \supset I_{n+1}$ for all $n \in \mathbb{N}$. Show that $\bigcap I_n \neq \phi$.*

*Solution.* Let $(x_n)$ be a sequence such that $x_n \in I_n$ for all $n \in \mathbb{N}$. Then $(x_n)$ is a bounded sequence as $x_n \in I_1$, which is bounded, for all $n \in \mathbb{N}$. It follows from the B-W Theorem that $(x_n)$ has a convergent subsequence $(y_n := x_{j(n)})$. Write $x := \lim y_n$. We show that $x \in \bigcap I_n$.
Suppose not. Then $x \in (\bigcap I_n)^c$, that is $x \in \bigcup I_n^c$. Therefore, there exists $N \in \mathbb{N}$ such that $x \notin I_N = [a_N, b_N]$. In otherwords, $x > b_N$ or $x < a_N$. Suppose $x > b_N$. This means $\lim y_n > b_N$. Therefore we have that $y_n > b_N$ eventually so $y_n \notin I_N$ eventually. Since $(I_n)$ is a nested interval, it means that $y_n \notin I_k$ for all $k \geq N$ eventually. However by construction of $(y_n)$ we have $y_n = x_{j(n)} \in I_{j(n)}$ for all $n \in \mathbb{N}$. Therefore for a large enough $M \in \mathbb{N}$ (where $j(M) \geq N$ and $y_M \notin I_k$ for all $k \geq N$), we must have $y_M = x_{j(M)} \in I_{j(M)}$ but $y_M \notin I_{j(M)}$. Hence, contradiction arises. The case for $\lim y_n > a_N$ is similar. Combining the two cases we have that $\lim y_n \in \bigcap I_n$ and so $\bigcap I_n \neq \phi$

**Example 2.3** (Diagonalization Argument)**.** *Let $(x_n)$ be a sequence of real numbers. Write $A := \{x_n : n \in \mathbb{N}\}$. For all $m \in \mathbb{N}$, let $f_m : A \to \mathbb{R}$ be a function. Suppose $f_m$ is bounded for all $m \in \mathbb{N}$.*

*a. Show that there exists a subsequence $(y_n)$ of $(x_n)$ such that $(f_1(y_n))$ and $(f_2(y_n))$ converges.*

*b. Fix $k \in \mathbb{N}$. Construct a subsequence $(y_n)$ of $(x_n)$ such that $(f_1(y_n)), \cdots (f_k(y_n))$ converges.*

*c. Show that there exists a subsequence $(y_n)$ of $(x_n)$ such that $(f_m(y_n))$ converges for all $m \in \mathbb{N}$.*

*Solution.*

a. Note that $(f_1(x_n))$ is a bounded sequence. By B-W Theorem, it has a convergent subsequence $(f_1(y_n^{(1)}))$. Note that then $(f_2(y_n^{(1)}))$ is a bounded sequence (not necessrily converging). By B-W theorem again, there exists a converging subsequence $(f_2(y_n^{(2)}))$ of $(f_2(y_n^{(1)}))$. Note that $(f_1(y_n^{(2)}))$ is then a subsequence of $(f_1(y_n^{(1)}))$, which also converges. Therefore the subsequence $(y_n^{(2)})$ is the required subsequence of $(x_n)$

b. Repeat the process in $(a)$ $k$ times. Then the subsequence $(y_n^{(k)})$ is the required one.

c. Define $y_n := y_n^{(n)}$ for all $n \in \mathbb{N}$ using the previous notations. It left as an exercise for the readers to check that $(y_n)$ is a subsequence of $(x_n)$ such that $\lim_n f_m(y_n)$ exists for all $m \in \mathbb{N}$.

**Quick Practice.**

1. *Let $(x_n)$ be a bounded sequence. Let $x \in \mathbb{R}$ such that every convergent subsequence of $(x_n)$ converges to $x$. Show that $(x_n)$ converges to $x$.*

2. *Let $A \subset \mathbb{R}$ be a subset. Recall that a point $x \in \mathbb{R}$ is called an **accumulation point** of $A$ if for all $\epsilon > 0$ there exists $a \in A$ where $a \neq x$ such that $|a - x| < \epsilon$.*

   (a) *Show that $x \in \mathbb{R}$ is an accumulation point of $A$ if and only if there exists a sequence $(a_n)$ in $A \backslash \{x\}$ such that $\lim a_n = x$.*

   (b) *Show that if $x$ is an accumulation point of $A$ then for all $\epsilon > 0$, there exists infinitely many points $a \in A \backslash \{x\}$ such that $|a - x| < \epsilon$*

   (c) *Let $(x_n)$ be a sequence and $A := \{x_n\}$ be its underlying set. Show that if $x$ is an accumulation point of $A$ then $x$ is a sequential cluster point of $(x_n)$ (see Definition 1.4).*

   (d) *Does the converse of part (c) holds for sequences in general?*

3. *Recall that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.*

   (a) *For all $r \in \mathbb{R}$, show that there exists a sequence $(q_n)$ of rational numbers converging to $r$.*

   (b) *Show that there exists a family of uncountably many subsets of $\mathbb{Q}$ such that intersections of any two of them has at most finitely many elements, that is, write $\{A_i\}_{i \in I}$ the collection of subsets with an uncountable index set $I$, then $A_i \cap A_j$ is a finite set for all $i \neq j$.*