1 Basic Applications of Subsequences

Definition 1.1. Let $x : \mathbb{N} \to \mathbb{R}$ be a sequence and $k : \mathbb{N} \to \mathbb{N}$ be a strictly increasing map. Then we call the sequence $y := (y_n := x_{k(n)})$ (in other words, $y = x \circ k$) a *subsequence* of x.

Theorem 1.2. *Let (x_n) be a sequence and (y_n) a subsequence. Suppose (x_n) converges. Then (y_n) converges and $\lim y_n = \lim x_n$.*

Example 1.3. *Show that $\lim x_n = \sqrt{e}$ where $x_n := (1 + \frac{1}{2n})^n$*

Solution. Consider the sequence $(z_n := (1 + \frac{1}{n})^n)$. Then $\lim z_n = e$. Now consider its subsequence $(y_n := z_{2n})$, that is $y_n = (1 + \frac{1}{2n})^{2n} = x_n^2$. By the consistency of subsequential limits, we have $\lim y_n = \lim z_n = e$. Therefore, $\lim x_n^2 = \lim y_n = e$. Note that since $x_n \geq 0$ for all $n \in \mathbb{N}$, we have (x_n^2) converges if and only if (x_n) converges (see Tutorial 3 Exercises). Therefore by algebraic property of limits, we have $(\lim x_n)^2 = e$ and so $\lim x_n = \sqrt{e}$ or $\lim x_n = -\sqrt{e}$ where the latter can be easily rejected.

Proposition 1.4. *Let (x_n) be a sequence. Suppose there exists two subsequences converging differently. Then (x_n) is divergent.*

Example 1.5. *Show that the sequence $(x_n := \cos(n\pi/3))$ is divergent.*

Solution. Consider the sequence $(y_n := x_{6n})$. Then it is easy to see that $y_n = \cos(2n\pi) = 0$ for all $n \in \mathbb{N}$. Hence, $\lim y_n = 0$. On the other hand, we consider the sequence $(z_n := x_{6n+1})$. Then $z_n = \cos(2n\pi + \pi/3) = 1/2$ for all $n \in \mathbb{N}$. Hence $\lim z_n = 1/2$. Since (y_n), (z_n) are subseqeunces converging differently, it follows that (x_n) does not converge.

It is important to note that subsequences arise naturally when considering divergent sequences.

Theorem 1.6. *Let (x_n) be a sequence and $x \in \mathbb{R}$. Then (x_n) does not converge to x if and only if there exists $\epsilon > 0$ such that there exists a subsequence (y_n) of (x_n) such that*

$$|y_n - x| \geq \epsilon$$

for all $n \in \mathbb{N}$

Example 1.7. *Let (x_n) be a sequence such that $0 \leq x_n \leq 1/n$ for all $n \in \mathbb{N}$. Show that $\lim x_n = 0$ using a contradiction argument.*

Solution. Suppose not. Then $\lim x_n \neq 0$. By subsequential characterization of divergence, there exists $\epsilon_0 > 0$ and a subsequence $(y_n := x_{k(n)})$ such that $\left|x_{k(n)}\right| \geq \epsilon_0$ for all $n \in \mathbb{N}$. It follows that we have $\frac{1}{k(n)} \geq \left|x_{k(n)}\right| \geq \epsilon_0$ for all $n \in \mathbb{N}$. This implies that $1/\epsilon_0 \geq k(n)$ for all $n \in \mathbb{N}$, showing that $\{k(n) : n \in \mathbb{N}\}$ is bounded in $\mathbb{R}$. Howerver $\{k(n) : n \in \mathbb{N}\} \subset \mathbb{N}$ is an infinite subset and is thus unbounded in $\mathbb{R}$ (why?). Contradiction arises. It follows that $\lim x_n = 0$.

Quick Practice.

1. *For each of the following sequences $x := (x_n)$, determine if they converge. If yes, find and verify their limits.*

   a) $x_n = (-1)^n$ b) $x_n = \frac{(-1)^n}{n}$ c) $x_n = \sin(2n\pi) + \cos(2n\pi)$

   d) $x_n = (1 + 1/n^2)^{n^2}$ e) $x_n = (1 + 1/3n)^n$ f) $x_n = (1 + 1/kn)^n$, $k \in \mathbb{N}$

2. *Let $x := (x_n)$ be a sequence and $y := (y_n)$ a subsequence of x. Furthermore, let $z := (z_n)$ be a subsequence of y (and so z is a sub-sub sequence of x). Suppose (x_n) is convergent.*

   (a) *Is it true that (z_n) converges?*

   (b) *If yes, do we have any information on $\lim z_n$?*

   Prove your assertions.

3. *Recall that $A \subset \mathbb{R}$ is a dense subset if and only if for all $\epsilon > 0$ and $r \in \mathbb{R}$, there exists $a \in A$ such that $|a - r| < \epsilon$. Let (x_n) be a sequence. Supppose the set $\{x_n : n \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

   (a) *Show that the subset $\{x_n : n \in \mathbb{N}, n \geq 5\}$ is dense.*

   (b) *Show that the subsets $A_k := \{x_n : n \in \mathbb{N}, n \geq k\}$ is dense for all $k \in \mathbb{N}$.*

   (c) *Show that for all $r \in \mathbb{R}$, there exists a subseqence (y_n) of (x_n) such that $\lim y_n = r$.*

2 The Bounded Monotone Convergence Theorem

Theorem 2.1 (Bounded Monotone Convergence of Sequences)**.** *Let (x_n) be a bounded above (resp. below) sequence that is increasing (resp. decreasing). Then (x_n) is convergent. Furthermore we have $\lim x_n = \sup\{x_n : n \in \mathbb{N}\}$ (resp. $\lim x_n = \inf\{x_n : n \in \mathbb{N}\}$).*

Example 2.2. *Consider $x := (x_n)$ where $x_n := 2^{1/n}$. Show that $\lim x_n = 1$.*

Solution. The limit can be verified using definitions together with the binomial theorem. We present here another argument using subsequences.
First we claim that (x_n) is a bounded below, decreasing sequence. It is bounded below because $x_n = 2^{1/n} \geq 1$ for all $n \in \mathbb{N}$ as $2 \geq 1$. To show that it is decreasing, it suffices to show that $\frac{x_n}{x_{n+1}} \geq 1$ for all $n \in \mathbb{N}$. This is true because we have

$$(\frac{x_n}{x_{n+1}})^{n+1} = (\frac{2^{1/n}}{2^{1/(n+1)}})^{n+1} = \frac{2^{\frac{n+1}{n}}}{2} = 2^{\frac{1}{n}} \geq 1$$

for all $n \geq \mathbb{N}$. It follows that

$$\frac{x_n}{x_{n+1}} \geq 1 \Longrightarrow x_n \geq x_{n+1}$$

for all $n \in \mathbb{N}$ and so (x_n) is decreasing. By Bounded Monotone Convergence, (x_n) is convergent.
Now write $x := \lim x_n$ and consider the subsequence $(y_n := x_{2n})$. Then we have $y_n = 2^{1/(2n)} = x_n^{1/2}$ for all $n \in \mathbb{N}$. By the consistency of subsequential limits, we have $\lim y_n = x$. However, we also have $\lim y_n = \lim x_n^{1/2} = x^{1/2}$ (why?). It then follows that $x = x^{1/2}$ and so we have $x = 0$ or $x = 1$. The former can be easily rejected (why?). Therefore it must be the case that $\lim x_n = x = 1$.

Quick Practice. *Find and verify the limits for each of the following sequences $x := (x_n)$.*

a) $x_n := 3^{1/n}$ *b)* $x_n := (1/3)^{1/n}$ *c)* $x_n := (1/2)^n$

d) $x_n := r^n, r \in (0,1)$ *e)* $x_n := a^{1/n}, a \geq 1$ *f)* $x_n := a^{1/n}, a \in (0,1)$

3 Limit Superior and Limit Inferior

Definition 3.1. Let (x_n) be a bounded sequence. Then

- The sequence (y_n) with $y_n := \sup_{k\geq n} x_k$ is bounded below decreasing. We call the limit $\lim y_n = \inf_n y_n$ the *limit superior* of (x_n) and denote it by $\overline{\lim}\, x_n$ or $\limsup x_n$.
- The sequence (z_n) with $z_n := \inf_{k\geq n} x_k$ is bounded above increasing. We call the limit $\lim z_n = \sup_n z_n$ the *limit inferior* of (x_n) and denote it by $\underline{\lim}\, x_n$ or $\liminf x_n$.

Theorem 3.2 (Lemma to B-W Theorem)**.** *Let (x_n) be a bounded sequence. Then there exists subsequences of (x_n) converging to $\limsup x_n$ and $\liminf x_n$.*

Example 3.3 (subsequence chararacterization of limsup)**.** *Let (x_n) be a bounded sequence. Write $x := \limsup x_n$. Show that x is the greatest subsequencetial limits of (x_n), that is, if $z \in \mathbb{R}$ is a limit of some subsequence of (x_n), then $z \leq x$.*

Solution. Let $(x_{k(n)})$ be some subsequence such that $\lim x_{k(n)} = z$. Note that $x_{k(n)} \leq \sup_{m\geq k(n)} x_m$ for all $n \in \mathbb{N}$ and $\lim_n \sup_{m\geq k(n)} x_m = \limsup x_n = x$ (why?). Therefore by order property of limits, we have $z = \lim x_{k(n)} \leq x$.

Example 3.4. *Let (x_n) be a bounded sequence. Write $x := \limsup x_n$. Let $r \in \mathbb{R}$. Suppose $r > x$. Show that there exists at most finitely many terms in (x_n) such that $r \leq x_n$.*

Solution. Suppose not. Then there exists infinitely many terms in (x_n) such that $r \leq x_n$. In other words, there exists a subsequence (y_n) of (x_n) such that $r \leq y_n$ for all $n \in \mathbb{N}$ (why?). Then by considering a subsequence (z_n) of (y_n) which converges (for example to $\limsup y_n$) and noticing that (z_n) is a subsequence of (x_n), we have $\lim z_n \leq \limsup x_n = x$ by subsequence characterization of limsup. However, by order limit property, we have $x < r \leq \lim z_n \leq x$. Contradiction arises.

Quick Practice. *Establish analogs of Example 3.3 and Example 3.4 for the limit inferior of a bounded sequence.*