1 Recall: The Natural Numbers

Theorem 1.1 (Archimedean Property). *Let $X = \mathbb{N} \subset \mathbb{R}$ be the set of natural numbers. Then X is not bounded above.*

Corollary 1.2 (ϵ- characterization of the Archimedean Property). *Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. In other words,* $\inf\{1/n : n \in \mathbb{N}\} = 0$.

Example 1.3. *Let $X := \{1/n^2 : n \in \mathbb{N}\}$. Show that* $\inf X = 0$.

Solution. First 0 is a lower bound of X clearly. It remains to show that 0 approximates X by the $\epsilon-$characterization of infimum. Let $\epsilon > 0$. Then by the Archimediean Property, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Since $N \geq 1$ as $N \in \mathbb{N}$, we have $N \leq N^2$ (why?). Therefore, it follows that $1/N^2 \leq 1/N < \epsilon = 0 + \epsilon$ and so 0 approximates X. It follows that $\inf X = 0$.

Quick Practice. *For each of the following subsets X, determine and explain whether* $\sup X$ *and* $\inf X$ *exist. If yes, find them.*

a) $X = \mathbb{Q}$ *b)* $X = \{1/n^3 : n \in \mathbb{N}\}$ *c)* $X = \{(2n+3)/n^3 : n \in \mathbb{N}\}$

2 Sequences

Definition 2.1 (Sequences). Let $x : \mathbb{N} \to \mathbb{R}$ be a function. Then we call x a sequence of real numbers.

Remark. For a sequence x, we usually denote its image by $x_n := x(n)$ for all $n \in \mathbb{N}$. We would also use (x_n) to denote the sequence (function) x and write "let $x := (x_n)$ be a sequence" if we have to define a sequence in the first place.

Example 2.2.

a. *A constant sequence $x := (x_n)$ is sequence that is a constant function, that is, there exists $c \in \mathbb{R}$ such that $x_n = c$ for all $n \in \mathbb{N}$*

b. *A bounded sequence $x := (x_n)$ is a sequence that is also a bounded function, that is, there exists $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.*

c. *A sequence is recursively defined if terms depend on previous terms. The Fibonacci sequence (f_n) is recursivelly defined by the relations*

$$f_1 = f_2 = 0 \qquad f_n = f_{n-2} + f_{n-1} \text{ for } n \geq 3$$

d. *The rational number $\mathbb{Q}$ is countably infinite. Therefore, we can write $\mathbb{Q} = (q_n)$ as a sequence by considering any bijection from $\mathbb{N}$ to $\mathbb{Q}$.*

Here comes the most important definition concerning sequences: their limits.

Definition 2.3 (Sequential imits). Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. We say that x is a limit of (x_n) if for all $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that for all $n \geq N$, we have

$$|x_n - x| < \epsilon$$

We call (x_n) a convergent sequence if it has a limit.

Remark.

- If a sequence (x_n) converges, then its limit is unique. We denote *the* limit as $\lim_n x_n$.
- The limit of a sequence is some point that the sequence gets close to *eventually.*

Example 2.4. *Consider the sequence $x := (x_n)$ with $x_n := 1/n$. Show that x converges.*

Solution. We can find an explicit limit of x. In fact we claim that $\lim 1/n = 0$. To this end, let $\epsilon > 0$. Then by Archimedean Property, there exists $N \in \mathbb{N}$ such that $1/N < \epsilon$. Now suppose $n \geq N$, we have $|1/n - 0| = 1/n \leq 1/N < \epsilon$ and so $\lim 1/n = 0$ by definition.

Quick Practice. *For each of the following sequences $x := (x_n)$, find and verifty their limits.*

a) $x_n := \pi$ *b)* $x_n := 1/n^2$ *c)* $x_n := 1 - 5/n$

d) $x_n := \dfrac{n}{n^2+1}$ *e)* $x_n := \sqrt{n+1} - \sqrt{n}$ *f)* $x_n := \dfrac{3n}{2n+9}$

g) $x_n := \dfrac{n^2-3}{4n^2-n-1}$ *h)* $x_n := \dfrac{4n-7}{2n-3}$ *i)* $x_n := n$

Definition 2.5 (Divergent Sequences)**.** Let (x_n) be a sequence. We say it diverges if it is not a convergent sequence. Equivalently, for all $x \in \mathbb{R}$, (x_n) does NOT converge to x.

Remark. To show that (x_n) does not converge to x is to verify the negation of sequential convergence. In other words, it is to show that there exists $\epsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $n_k \geq k$ such that $|x_{n_k} - x| \geq \epsilon$

Example 2.6. *Show that the sequence $(x_n := (-1)^n)$ does not converge to ± 1.*

Solution. We first show that x_n does not converge to 1. To this end, we take $\epsilon := 1$. Moreover, we take $n_k := 2k + 1 \geq k$ for all $k \in \mathbb{N}$. Then $x_{n_k} = -1$ for all $k \in \mathbb{N}$. Therefore $|x_{n_k-1}| = |-1-1| = 2 \geq 1$ for all $k \in \mathbb{N}$. By definition, x_n does not converge to 1. The case of -1 is left to the readers.

Quick Practice. *Consider the sequence (x_n) where $x_n = 0$ when n even and $x_n = 1$ when n is odd.*

a. Show that the sequence does not converge to 0.

b. Show that the sequence does not converge to any $r \in \mathbb{R}$

3 Exercises

1. Let $x := (x_n)$ and $y := (y_n)$ be sequences of real numbers such that $y_n := x_{2n}$
   (a) Show that if (x_n) converges, then (y_n) converges.
   (b) Find an example such that the converse does not hold.
2. Recall that a subset $A \subset \mathbb{R}$ is *dense* if and only if for all open intervals (a, b) with $a < b$, we have $A \cap (a, b) \neq \phi$.
   (a) Show that a subset A is dense if and only if for all $r \in \mathbb{R}$ and $\epsilon > 0$, there exists $q \in A$ such that $|q - r| < \epsilon$. This is the $\epsilon-$*characterization for dense subset*s.
   (b) Hence, show that a subset A is dense if and only if for all $r \in \mathbb{R}$, there exists a sequence (a_n) where $a_n \in A$ such that $\lim a_n = r$.
3. For each of the following sequences $x := (x_n)$, determine if it convergences or not. If it converges, find its limit using definitions; otherwise, make a proof on why it diverges.
   a) $x_n := (-1)^n/n$
   b) $x_n := \cos(n\pi)$
   c) $x_n := \sqrt{n}$
   d) $x_n := \sqrt{2}$, corrected to nth decimal place
4. We call a sequence $x := (x_n)$ finitely supported if $x_n = 0$ except for finitely many $n \in \mathbb{N}$. Show that x is a convergent sequence.

   We would be investigating the algebraic properties of sequences in Q5 - 7.
5. Let $s := \{x : \mathbb{N} \to \mathbb{R}\}$ be the set of all real valued sequences, For $x, y \in s$, define $x + y$ such that $(x + y)(n) := x(n) + y(n)$. For $\alpha \in \mathbb{R}$ and $x \in s$, define $\alpha \cdot x$ such that $(\alpha \cdot x)(n) := \alpha x(n)$ for all $n \in \mathbb{N}$.
   a) Show that $(s, +, \cdot)$ is a vector space.
   b) Show that there exists an infinite subset of linearly independent elements in $(s, +, \cdot)$.
6. Let $b := \{x : \mathbb{N} \to \mathbb{R} \mid x \text{ is bounded}\}$ be the set of bounded sequences and $c := \{x : \mathbb{N} \to \mathbb{R} \mid x \text{ is convergent}\}$ be the set of convergent sequences.
   (a) Verify the following (proper) subset inclusions:
   $$c \subsetneq b \subsetneq s$$
   where s is the vector space of real sequences defined in Q5.
   (b) Show that b, c are infinite dimensional vector sub**spaces** of s.
   (c) Define $T : c \to \mathbb{R}$ by $Tx := \lim x_n$ where $x := (x_n)$. Show that T is linear.
7. For all $r \in \mathbb{R}$, define $c_r := \{x := (x_n) \in c \mid \lim x_n = r\}$. Show that there is a unique $r \in \mathbb{R}$ such that c_r is a vector space and hence a vector subspace of c.