MATH 2058 - Revision Test 1 - Solutions

1 (10 marks)**.** Suppose $\lim a_n = 3$. Show by using definitions that

$$\lim_{n\to\infty} \frac{a_n^2+1}{a_n-2} = 10$$

Solution. Let $1/2 > \epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|a_n - 3| < \epsilon/11$ for all $n \geq N$ as $\lim a_n = 3$. Now we approximate the distance inequality concerning the limit in the question: for all $n \in \mathbb{N}$, we have

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = \left|\frac{a_n^2 - 10a_n + 21}{a_n - 2}\right| = |a_n - 3|\left|\frac{a_n-7}{a_n-2}\right| = |a_n-3|\underbrace{\left|1 - \frac{5}{|a_n-2|}\right|}_{:=(I)}$$

In addition we have for all $n \geq N$ that

$$(I) := \left|1 - \frac{5}{|a_n-2|}\right| \leq 1 + \frac{5}{|a_n-2|} \leq 1 + \frac{5}{|2-3| - |a_n-3|} \leq 1 + \frac{5}{1-|a_n-3|} \leq 1 + \frac{5}{1-1/2} = 11$$

by multiple uses of the triangle inequality. Hence we have for all $n \geq N$ that

$$\left|\frac{a_n^2+1}{a_n-2} - 10\right| = |a_n-3| \cdot (I) \leq \frac{\epsilon}{11} \cdot 11 = \epsilon$$

This proves the limit in the question.

2 (10 marks)**.** Let (x_n) be a sequence. Suppose $\lim(-1)^n x_n = 0$. Is it true that (x_n) converges? Prove your assertion and find the limit **only** if it converges.

Solution. Yes, (x_n) converges. Let $\epsilon > 0$. Since $\lim(-1)^n x_n = 0$, it follows that there exists $N \in \mathbb{N}$ such that $|(-1)^n x_n| < \epsilon$ for all $n \geq N$. Note that in fact we have $|x_n - 0| = |(-1)^n x_n|$ for all $n \in \mathbb{N}$. Combining this with the previous approximation, we obtain that for all $n \geq N$ that $|x_n| < \epsilon$. This prove $\lim x_n = 0$

3 (10 marks)**.** Let (x_n) be a sequence. Let $a > 0$ be such that $x_1 > \sqrt{a}$. Suppose that (x_n) satisfies the recursive relation

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$$

for all $n \geq 1$. Show that (x_n) converges and find its limit.

Solution. We first show that $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$ by induction. Note $x_1 \geq \sqrt{a}$ is given. Now suppose $x_k \geq \sqrt{a}$ for some $k \in \mathbb{N}$ then we have

$$x_{k+1} - \sqrt{a} = \frac{1}{2}(x_k + \frac{a}{x_k}) - \sqrt{a} = \frac{1}{2x_k}(x_k^2 - 2\sqrt{a}x_k + a) = \frac{1}{2x_k}(x - \sqrt{a})^2 \geq 0$$

as $x_k \geq \sqrt{a} > 0$. It follows that $x_{k+1} \geq \sqrt{a}$. Hence $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$ by induction, implying (x_n) is bounded below (and well-defined since the fraction in the recursive relation is).
Next, we claim that (x_n) is decreasing. Fix $n \in \mathbb{N}$. Then we have

$$\frac{x_{n+1}}{x_n} = \frac{1}{2}(1 + \frac{a}{x_n^2}) \leq \frac{1}{2}(1 + \frac{a}{\sqrt{a}^2}) = \frac{1}{2}(1+1) = 1$$

by the claim that $x_n \geq \sqrt{a}$ for all $n \in \mathbb{N}$. It follows that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$ and so (x_n) is decreasing.
To conclude, (x_n) is bounded below decreasing and so (x_n) is convergent by the bounded monotone convergence theorem.

Next, we find the limit of (x_n). Write $x := \lim x_n$ and note that $\lim x_{n+1} = \lim x_n = x$ since (x_{n+1}) is a (tail) subsequence. By rewriting the recursive relation it follows that we have

$$x^2 = \lim x_{n+1} \lim x_n = \lim_n x_{n+1}x_n = \lim \frac{1}{2}(x_n x_n + a) = \frac{1}{2}(\lim x_n \lim x_n + a) = \frac{1}{2}(x^2 + a)$$

It follows that $x^2 = a$ and so $x = \sqrt{a}$ or $x = -\sqrt{a}$. The latter can be rejected since $x_n \geq 0$ for all $n \in \mathbb{N}$.

4 (20 marks)**.** Let (x_n) be a bounded sequence. We call $x \in \mathbb{R}$ a sequential cluster point of (x_n) if for all $\epsilon > 0$ and for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|x_n - x| < \epsilon$. Define

$$E := \{x \in \mathbb{R} : x \text{ a sequential cluster point of } (x_n)\}$$

i. Show that E is non-empty.

ii. Show that E is a singleton if and only if (x_n) converges.

Solution.

1. Since (x_n) is bounded, by $B - W$ theorem, there exists a convergence subsequence $(x_{k(n)})$ where $k : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Write $y := \lim x_{k(n)}$. we claim that $y \in E$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\left|x_{k(n)} - y\right| < \epsilon$ for all $n \geq N$ as $\lim x_{k(n)} = y$. Now pick $j \in \mathbb{N}$, suppose $j < N$, it follows that $k(N) \geq N \geq j$ such that $\left|x_{k(N)} - y\right| < \epsilon$. On the other hand, suppose $j \geq N$, we have that $k(j) \geq j$ and $\left|x_{k(j)} - y\right| < \epsilon$. By definition, y is a sequential cluster point.

2. Let B be the set of subsequential limits of (x_n). The proof in Q1 in fact showed that $B \subset E$. We proceed with the observation.
   $(\Rightarrow)$. Suppose E is a singleton. Since $B \subset E$ and B is non-empty by the B-W theorem, it follows that B is a singleton. Write $B = \{x\}$ then by definition of B every convergent subsequence converges to x. We clain that (x_n) converges to x. Suppose not. Then there existed $\epsilon_0 > 0$ and a subsequence (y_n) of (x_n) such that $|y_n - x| \geq \epsilon_0$. Since (y_n) is in turn bounded, it has a convergent subsequence (z_n), which is also a subsequence of (x_n). Hence by B being a singleton we have that $\lim z_n = x$. However, we also have $|z_n - x| \geq \epsilon_0$ for all $n \in \mathbb{N}$. It follows that contradiction arises as $0 = \lim |z_n - x| \geq \epsilon_0$. It must be the case that $\lim x_n = x$.

   $(\Leftarrow)$. Now suppose (x_n) converges. We show that E is a singleton. Write $x := \lim x_n$. As (x_n) is a subsequence of itself, it follows that $x \in B \subset E$ (where B is the set of subsequential limits of (x_n)) by Q1. It remains to show that $E \subset \{x\}$. Suppose $y \in E$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$ as $\lim x_n = x$. Moreover by definition of a sequential cluster point, it follows that there exists $k(N) \geq N$ such that $\left|y - x_{k(N)}\right| < \epsilon$. Hence by the triangle inequality, we have

   $$|x - y| \leq \left|x - x_{k(N)}\right| + \left|x_{k(N)} - y\right| \leq \epsilon + \epsilon = 2\epsilon$$

   Since the choice of ϵ is arbitrary, it follows that $|x - y| \leq 0$ and so $|x - y| = 0$. This imples $x = y$. Therefore $E = \{x\}$ is a singleton.

   Altenative solution: We can do this question with the help of liminf and limsup. From the Tutorial 4, we have shown that sequential cluster points are subsequential limits. Therefore in fact we have $B = E$ where B is the set of subsequential limits.
   $(\Rightarrow)$. Suppose E is a singleton. Write $E = \{x\}$ where $x \in \mathbb{R}$. Then x is the only subsequential limit of (x_n). Note that by results in the lecture notes that $\limsup x_n$ and $\liminf x_n$ are subsequential limits. Therefore $\limsup x_n = \liminf x_n = x$. It follows from the characterization of sequential convergence using limsup and liminf that (x_n) converges with $\lim x_n = x$.

   $(\Leftarrow)$. Suppose that (x_n) converges. Then $\liminf x_n = \limsup x_n$. Note that from Tutorial 3, we saw that $\limsup x_n$ is the greatest subsequential limit while $\liminf x_n$ is the minimum subsequential limit. In other words, $\liminf x_n = \inf E$ and $\limsup x_n = \sup E$ by identifying E with the set of subsequential limits. Therefore, $\sup E = \inf E$, implying the E has at most 1 element. By part (i), we have seen that E is non-empty. Therefore E is a singleton.