MATH 2058 - HW 8 - Solutions

1 (P. 148 Q2)**.** Let $f(x) := 1/x^2$. Show that

1. f is uniformly continuous on $A := [1, \infty)$
2. f is not uniformly continuous on $B := (0, \infty)$

Solution.

i. Let $\epsilon > 0$. Let $x, y \in A$. Then

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{x^2 - y^2}{x^2y^2}\right| = |x-y|\left|\frac{x+y}{x^2y^2}\right| \leq |x-y|\left|\frac{1}{x} + \frac{1}{y}\right| \leq |x-y||1+1| = 2|x-y|$$

Hence if we take $\delta := \epsilon/2 > 0$. Then for all $x, y \in A$ with $|x - y| < \delta$, we have $|f(x) - f(y)| \leq 2|x - y| \leq 2\delta < \epsilon$. It follows from definition that f is uniformly continuous on A.

ii. Define $x_n := 1/\sqrt{n}$ and $y_n := 1/n$ for all $n \in \mathbb{N}$. Then (x_n) and (y_n) are sequences in B. Then $\lim |x_n - y_n| = \lim |1/\sqrt{n} - 1/n| = 0$. However, for all $n \geq 2$, we have $|f(x_n) - f(y_n)| = \left|n - n^2\right| = |n||n-1| \geq n$. It follows that $\lim |f(x_n) - f(y_n)| = \infty$. It follows from the divergence criteria of uniform continuity that f is not continuous on B

2 (P. 148 Q6)**.** Let $f, g : A \to \mathbb{R}$ be uniformly continuous on A. Suppose they are both bounded, show that fg is uniformly continuous on A.

Solution. Let $\epsilon > 0$. Then there exists $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |f(x) - f(y)| &< \epsilon & \qquad |x - y| &< \delta_1 \\ |g(x) - g(y)| &< \epsilon & \qquad |x - y| &< \delta_2 \end{aligned}$$

Now take $\delta := \min\{\delta_1, \delta_2\} \leq \delta_1, \delta_2$. Then for all $x, y < \delta$, we have

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\leq (\sup |f(x) : x \in A| + \sup |g(x) : x \in A|)\epsilon \end{aligned}$$

It follows that fg is uniformly continuous on A.

3 (P. 148 Q7)**.** Let $f(x) := x$ and $g(x) := \sin x$. Show that

i. f and g are uniformly continuous on $\mathbb{R}$

ii. Show that fg is not uniformly continuous on $\mathbb{R}$

Solution.

i. We first show f is uniformly continuous on $\mathbb{R}$: let $\epsilon > 0$. Then take $\delta := \epsilon$. It follows that if $x, y \in \mathbb{R}$ with $|x - y| < \delta$ then $|f(x) - f(y)| = |x - y| < \delta = \epsilon$.
Next, we show that g is uniformly continuous on $\mathbb{R}$. Here we use the fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$. It follows that

$$|g(x) - g(y)| = |\sin x - \sin y| = \left|2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right| \leq \left|2\sin\left(\frac{x-y}{2}\right)\right| \leq 2\frac{|x-y|}{2} = |x-y|$$

Hence, g is a 1-Lipschitz function on $\mathbb{R}$. It follows that g is uniformly continuous.

ii. Take $x_n := 2n\pi$ and $y_n := 2n\pi + 1/n$. Then $fg(x_n) = 0$ while $fg(y_n) = (2n\pi + 1/n)\sin(1/n)$ for all $n \in \mathbb{N}$. Note that by the limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we have that $\lim_n \frac{\sin 1/n}{1/n} = 1$ using sequential criteria. It follows that

$$\lim fg(y_n) = \lim 2n\pi \sin(1/n) + \lim 1/n \sin(1/n) = 2\pi + 0 = 2\pi$$

Hence, we have $\lim |x_n - y_n| = |1/n| = 0$, but $\lim |fg(x_n) - fg(y_n)| = \lim |0 - fg(y_n)| = \lim |fg(y_n)| = 2\pi$ It follows from the divergence criteria for uniform continuous that fg is not uniform continuous on $\mathbb{R}$.