MATH 2058 - HW 7 - Solutions

1 (P. 129 Q9)**.** Let $A \subset B \subset \mathbb{R}$ be subsets of $\mathbb{R}$. Let $f : B \to \mathbb{R}$ be a function and $g : A \to \mathbb{R}$ be the restriction of f on A, that is, $g(x) = f(x)$ for all $a \in A$.

i. Show that if f is continuous at $c \in A$, then g is continuous at c.

ii. Give an example to illustrate that if g is continuous at $c \in A$, it is not necessary that f is continuous at c.

Solution.

i. **Method 1: By definition.** Let $\epsilon > 0$. Then by the continuity of f at $c \in A$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in B$ with $|x - c| < \delta$. Now pick $x \in A$ with $|x - c| < \delta$. Since $A \subset B$, we have $x \in B$ and so $|g(x) - g(c)| = |f(x) - f(c)| < \epsilon$. It follows from definition that g is continuous at $c0$

   Method 2: Sequential criteria. Let (x_n) be a sequence in A such that $\lim x_n = c \in A$. Then $\lim f(x_n) = f(c)$ as f is continuous at c as (x_n) is a sequence in $A \subset B$. Note that $f(x_n) = g(x_n)$ for all $n \in \mathbb{N}$ and $f(c) = g(c)$ as $x_n \in A$ for all $n \in \mathbb{N}$ and $c \in A$. It follows that $\lim g(x_n) = \lim f(x_n) = f(c) = g(c)$. Hence, g is continuous at $c \in A$.

ii. Consider $A = \{0\}$ and $B := \mathbb{R}$. Then $A \subset B \subset \mathbb{R}$. Consider $f : B \to \mathbb{R}$ where $f = \chi_{\mathbb{Q}}$, the characteristic function of $\mathbb{Q}$, or the Dirichlet function. Then we have shown in Tutorials that f is not continuous everywhere on $\mathbb{R}$. Nonetheless, the restriction of f on A is continuous since A is a singleton (in fact every function having a finite domain is continuous (why?)) .

2 (P. 140 Q7)**.** Consider the equation

$$x = \cos x$$

i. Show that the equation has a solution on the intevral $[0, \pi/2]$

ii. Using the Bisection Method and a calculator, find an approximate solution to the equation with error less than 10^{-3}

Solution.

i. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x - \cos(x)$. Then $x \in [0, \pi/2]$ is a solution to $x = \cos x$ if and only if $f(x) = 0$. It suffices to find a zero of f on $[0, \pi/2]$. Note that $f(0) = 0 - \cos(0) = -1$ and $f(\pi/2) = \pi/2 - \cos(\pi/2) = \pi/2$. Since f is continuous on $\mathbb{R}$, in particular, on $[0, \pi/2]$, which is an interval, and $f(0) < 0 < f(\pi/2)$, it follows from the Intermediate Value Theorem (IVT) that $f(z) = 0$ for some $z \in [0, \pi/2]$.

ii. Let $\epsilon_n > 0$ be the error of the nth iteration in the bisection method for all $n \in \mathbb{R}$ (we consider the initial consition to be the zeroth interation). Then it is clear that $\epsilon_n < \pi/2 \cdot 1/2^n$ for all $n \in \mathbb{N}$. Hence, for the error to be less than 10^{-3}, it suffices to consider 11 iterations. We present the result in the form of a table.

Bisection Method for a root of $f(x) = x - \cos(x)$ on $[0, \pi/2]$

n	Left (a_n)	Right (b_n)	Midpoints (c_n)	$\lvert a_n - b_n\rvert$	f(a_n)	f(b_n)	f(c_n)	sign of f(c_n)
0	0	1.570796	0.785398	1.570796	-1	1.570796	0.078291	+
1	0	0.785398	0.392699	0.785398	-1	0.078291	-0.531180	-
2	0.392699	0.785398	0.589049	0.392699	-0.531180	0.078291	-0.242421	-
3	0.589049	0.785398	0.687223	0.196350	-0.242421	0.078291	-0.085787	-
4	0.687223	0.785398	0.736311	0.098175	-0.085787	0.078291	-0.004640	-
5	0.736311	0.785398	0.760854	0.049087	-0.004640	0.078291	0.036607	+
6	0.736311	0.760854	0.748583	0.024544	-0.004640	0.036607	0.015928	+
7	0.736311	0.748583	0.742447	0.012272	-0.004640	0.015928	0.005630	+
8	0.736311	0.742447	0.739379	0.006136	-0.004640	0.005630	0.000491	+
9	0.736311	0.739379	0.737845	0.003068	-0.004640	0.000491	-0.002075	-
10	0.737845	0.739379	0.738612	0.001534	-0.002075	0.000491	-0.000792	-
11	0.738612	0.739379	0.738995	0.000767	-0.000792	0.000491	-0.000150	-

Table 1

It follows that any $x \in [a_{11}, b_{11}] = [0.738612, 0.739379]$ is an approximate solution with error less than 10^{-3}, in fact, less than 0.000767.