MATH 2058 - HW 5 - Solutions

1 (P.91 Q3)**.** Show directly from the definition that the following sequences (x_n) are not Cauchy sequences.

a) $x_n := (-1)^n$ b) $x_n := n + \frac{(-1)^n}{n}$ c) $x_n := \log n$

a. Take $\epsilon_0 := 1$. Pick $n \in \mathbb{N}$. Take $k(n) = 2n$ and $j(n) = 2n+1$ for all $n \in \mathbb{N}$. Then $k(n) \geq n$ and $j(n) \geq n$ and

$$\left|x_{k(n)} - x_{j(n)}\right| = \left|(-1)^{2n} - (-1)^{2n+1}\right| = 2 \geq 1$$

It follows from the negation of the definition that (x_n) does not converge.

b. Pick $n \in \mathbb{N}$. Take $k(n) := 4n$ and $j(n) := 2n$ for all $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, we have $k(n), j(n) \geq n$ and

$$\left|x_{k(n)} - x_{j(n)}\right| = \left|4n + \frac{1}{4n} - 2n - \frac{1}{2n}\right| = \left|2n - \frac{1}{4n}\right| \overset{(\star)}{=} 2n - \frac{1}{4n} \geq 2 - \frac{1}{4} \geq 1$$

The $(\star)$ follows because $2n \geq 1/4n$ for all $n \geq 1$. It follows from the negation of the definition that (x_n) does not converge.

c. Note that the natural base satisfies that $0 < e < 3$. Since the logarithmic is a strictly increasing, it follows that we have $1 < \log 3$. Now for all $n \in \mathbb{N}$, we pick $k(n) := 3n$ and $j(n) := n$. Then it follows that we have

$$\left|x_{k(n)} - x_{j(n)}\right| = |\log 3n - \log n| = |\log 3| \overset{(\star)}{=} \log 3 \geq 1$$

It follows from the negation of the definition that (x_n) does not converge.

Remark. You have to remove the absolute sign before comparing numbers (like the steps in $(\star)$) unless you are using triangle inequalities.

2 (P.91 Q9)**.** Let $r \in (0,1)$. Let (x_n) be a sequence such that $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$. Show that (x_n) is a Cauchy sequence.

Solution. Note that $\lim r^n = 0$ as $r \in (0,1)$. Let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|r^n| < (1-r)\epsilon$ for all $n \geq N$. Now pick $m, n \geq N$ such that $m > n$. Then we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n+1} + x_{n+1} - \cdots - x_{m-1} + x_{m-1} - x_m| \\ &\leq |x_n - x_{n+1}| + \cdots + |x_{m-1} - x_m| \\ &\leq r^n + \cdots + r^{m-1} \\ &= r^n(1 + \cdots r^{m-n-1}) \\ &= r^n \frac{1 - r^{m-n}}{1-r} \end{aligned}$$

by both the summation of geometric series and triangle inequalities. Finally note that (r^n) is strictly decreasing and $r^n \in (0,1)$ for all $n \in \mathbb{N}$. This implies that

$$|x_n - x_m| \leq r^n \frac{1 - r^{m-n}}{1-r} \leq r^n \frac{1}{1-r} \leq r^N \frac{1}{1-r} < (1-r)\epsilon \frac{1}{1-r} = \epsilon$$

for all $n, m \geq N$

Remark. Please try to address the main assumptions in questions ($r \in (0,1)$ this time) whenever giving solutions. Otherwise, marks may be deducted.