MMAT 5220 Complex Analysis and Its Applications

Lecture 7

§ Power series (cont'd)

Recall the

Lemma If $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ converges for $z_1 \neq z_0$, then $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent for $|z-z_0| < |z_1 - z_0|$.

Thm 1 Let $R>0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then

(1) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is absolutely convergent for $|z-z_0|<R$.

(2) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges for $|z-z_0|>R$.

Pf : (1) If $|z-z_0|<R$, $\exists\, z_1$ s.t. $|z-z_0|<|z_1-z_0|<R$.

Then $\sum_{n=0}^{\infty} a_n(z_1-z_0)^n$ converges, so the result follows from the Lemma.

(2) Suppose not. Then $\exists\, z_2$ s.t. $|z_2-z_0|>R$ but $\sum_{n=0}^{\infty} a_n(z_2-z_0)^n$ converges.

But then for $|z-z_0|<\frac{1}{2}(R+|z_2-z_0|)<|z_2-z_0|$, the Lemma implies that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges

Since $\frac{1}{2}(R+|z_2-z_0|)>R$, this contradicts the definition of R. #

Thm 2 Let $R>0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then

(1) For $0<R_1<R$, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniformly convergent for $|z-z_0|\leq R_1$.

(2) Hence $S(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ defines a continuous function for $|z-z_0|<R$.

Rmk $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is uniformly convergent for $|z-z_0|\leq R_1$ means

$\forall \varepsilon>0$, $\exists N_\varepsilon$ (indept of z) s.t. $\left|\sum_{n=N}^{\infty} a_n(z-z_0)^n\right|<\varepsilon$ $\forall N\geq N_\varepsilon$ & $\forall |z-z_0|\leq R_1$.

<u>Pf</u>: (1) Pick any z_1 s.t. $|z_1 - z_0| = R_1 < R$. Thm 1 says that $\sum_{n=0}^{\infty} a_n(z_1 - z_0)^n$ is absolutely convergent, i.e. $\sum_{n=0}^{\infty} |a_n||z_1 - z_0|^n$ converges.

So $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. $\sum_{n=N}^{\infty} |a_n| R_1^n = \sum_{n=N}^{\infty} |a_n||z_1 - z_0|^n < \varepsilon \quad \forall N \geq N_\varepsilon$.

$\Rightarrow \left| \sum_{n=N}^{\infty} a_n (z - z_0)^n \right| \leq \sum_{n=N}^{\infty} |a_n| R_1^n < \varepsilon \quad \forall N \geq N_\varepsilon$ and $\forall |z - z_0| \leq R_1$.

(2) Let $z^* \in B(z_0, R)$, i.e. $|z^* - z_0| < R$.

We want to show that $S(z)$ is continuous at z^*.

Let $\varepsilon > 0$. Choose R_1 s.t. $|z^* - z_0| < R_1 < R$.

Then part (1) says that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ uniformly converges on $\overline{B(z_0, R_1)}$

$\Rightarrow \exists N_\varepsilon$ s.t. $\left| \sum_{n=N}^{\infty} a_n (z - z_0)^n \right| < \frac{\varepsilon}{3} \quad \forall N \geq N_\varepsilon$ and $\forall |z - z_0| \leq R_1$.

Since a polynomial is continuous, $\exists \delta > 0$ s.t.

$$\left| \sum_{n=0}^{N_\varepsilon} a_n (z - z_0)^n - \sum_{n=0}^{N_\varepsilon} a_n (z^* - z_0)^n \right| < \frac{\varepsilon}{3} \quad \forall\, |z - z^*| < \delta .$$

Hence, $\forall\, |z - z^*| < \delta$,

$$\begin{aligned} |S(z) - S(z^*)| &\leq \left| S(z) - \sum_{n=0}^{N_\varepsilon} a_n (z - z_0)^n \right| \\ &\quad + \left| \sum_{n=0}^{N_\varepsilon} a_n (z - z_0)^n - \sum_{n=0}^{N_\varepsilon} a_n (z^* - z_0)^n \right| \\ &\quad + \left| S(z^*) - \sum_{n=0}^{N_\varepsilon} a_n (z^* - z_0)^n \right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \qquad \# \end{aligned}$$

<u>Thm 3</u> Let $R>0$ be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then

(1) (Term-by-term integration) For any contour γ in $B(z_0,R)$ and any continuous function g on γ, we have

$$\int_\gamma g(z)\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)dz = \sum_{n=0}^{\infty} a_n \int_\gamma g(z)(z-z_0)^n\,dz$$

(2) $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is analytic and

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \text{ in } B(z_0,R).$$ (Term-by-term differentiation)

Pf: (1) Let $M = \max_{z \in \gamma} |g(z)|$ and $L =$ length of γ.

$\exists\, 0 < R_1 < R$ s.t. γ is contained inside $\overline{B(z_0, R_1)}$.

Uniform convergence of $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ on $\overline{B(z_0,R_1)}$ implies that

$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon > 0$ s.t. $\left| \sum_{n=N+1}^{\infty} a_n (z-z_0)^n \right| < \varepsilon \quad \forall\, N+1 \geq N_\varepsilon$ and $\forall\, |z-z_0| < R_1$.

$$\Rightarrow \left| \int_\gamma g(z) \left(\sum_{n=0}^{\infty} a_n (z-z_0)^n \right) dz - \sum_{n=0}^{N} a_n \int_\gamma g(z)(z-z_0)^n dz \right| = \left| \int_\gamma g(z) \left(\sum_{n=N+1}^{\infty} a_n (z-z_0)^n \right) dz \right|.$$

$$\leq M \cdot L \cdot \varepsilon \quad \forall\, N+1 \geq N_\varepsilon$$

The result follows.

(2) Applying part (1) to $g \equiv 1$ and any closed contour $\gamma \subset B(z_0, R)$

$$\Rightarrow \int_\gamma S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_\gamma (z-z_0)^n\,dz = 0$$

So $S(z)$ is analytic by a previous thm.

Now for $z \in B(z_0,R)$, $\exists R_1 > 0$ s.t. $|z-z_0| < R_1 < R$. Take $\gamma = \{z \in \mathbb{C} : |z-z_0| = R_1\}$.

Then by the Cauchy integral formula and part (1), we have

$$S'(z) = \frac{1}{2\pi i}\int_\gamma \frac{S(w)}{(w-z)^2}\,dw = \sum_{n=0}^{\infty} \frac{a_n}{2\pi i}\int_\gamma \frac{(w-z_0)^n}{(w-z)^2}\,dw$$

$$= \sum_{n=0}^{\infty} a_n \left[\frac{d}{dw}(w-z_0)^n \Big|_{w=z}\right]$$

$$= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}. \quad \#$$

By part(2) of Thm 3 and induction, we have

Cor Let $R>0$ be the radius of convergence of $S(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

Then $a_n = \frac{1}{n!} S^{(n)}(z_0)$, i.e. $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is the Taylor series expansion of $S(z)$.

Rmk Thms 1–3 can be extended to Laurent series:

By setting $w=(z-z_0)^{-1}$ and $b_n = a_{-n}$, we have

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n} = \sum_{n=1}^{\infty} b_n w^n$$

and $\sum_{n=1}^{\infty} b_n w^n$ converges for $|w|<r$ means $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converges for $|z-z_0| > R_1 := \frac{1}{r}$.

e.g. • $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$

$$= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z^2}\right)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} \quad \text{for } 0<|z|<\infty$$

• $f(z) = \frac{\sinh z}{1+z} = (\sinh z)\left(\frac{1}{1+z}\right)$

$$= \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right)$$

$$= z - z^2 + z^3 - z^4 + \frac{z^3}{3!} - \frac{z^4}{3!} + \cdots \quad (\text{up to } z^4)$$

$$= z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \cdots \quad \text{for } |z|<1$$

- $$\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} = \frac{1}{z\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$= \frac{1}{z}\left(1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)^2 + \cdots\right)$$

$$= \frac{1}{z}\left(1 - \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \cdots\right)$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{360} z^3 + \cdots \quad \text{for } 0 < |z| < \pi \quad (\text{up to } z^3)$$

- $$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\Rightarrow \frac{d}{dz} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (2n+1) \cdot z^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z$$

- $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z|<1$$

$$\Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n \quad \text{for } |z|<1$$

$$\Rightarrow \frac{2}{(1-z)^3} = \sum_{n=1}^{\infty} (n+1) n z^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \quad \text{for } |z|<1$$

- $$f(z) := \begin{cases} \dfrac{\sin z}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$$

is entire because $\dfrac{\sin z}{z} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$ is convergent $\forall z \in \mathbb{C}$ and the value of the RHS at $z=0$ is 1.

§ Zeros and uniqueness of analytic functions

Def Suppose f is analytic at z_0. We say that f has a **zero of order m** at z_0 if $\exists$ a +ve integer m s.t. $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

Thm Let f be analytic in $|z - z_0| < R$. Then f has a zero of order m at z_0 iff $\exists$ analytic g s.t. $f(z) = (z - z_0)^m g(z)$ in $|z - z_0| < R$ and $g(z_0) \neq 0$.

Pf : ($\Longrightarrow$) Taylor's expansion of f around z_0 is given by

$$f(z) = \cancelto{0}{f(z_0)} + \cancelto{0}{f'(z_0)}(z - z_0) + \frac{\cancelto{0}{f^{(2)}(z_0)}}{2}(z - z_0)^2 + \cdots + \frac{\cancelto{0}{f^{(m-1)}(z_0)}}{(m-1)!}(z - z_0)^{m-1} + \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \cdots$$

$$= (z - z_0)^m \left[\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z - z_0) + \cdots \right]$$

So $f(z) = (z-z_0)^m g(z)$, where $g(z) := \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \cdots$

is analytic and $g(z_0) = f^{(m)}(z_0)/m! \neq 0$.

($\Leftarrow$) By uniqueness, the Taylor series expansion of f around z_0 is given by

$$f(z) = (z-z_0)^m g(z) = (z-z_0)^m \left[g(z_0) + g'(z_0)(z-z_0) + \cdots \right]$$
$$= g(z_0)(z-z_0)^m + g'(z_0)(z-z_0)^{m+1} + \cdots$$

This implies that $f(z_0) = f'(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) = m!\, g(z_0) \neq 0$. #