THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Week 6 Examples

1. Find the Laurent series for $1/z^2(1-z)$ in the regions:
   (a) $0 < |z| < 1$; (b) $|z| > 1$.

   Solution.

   (a) Recall that
   $$1/(1-z) = 1 + z + z^2 + z^3 + \cdots \quad \text{for } |z| < 1.$$
   Hence,
   $$1/z^2(1-z) = \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad \text{for } 0 < |z| < 1.$$

   (b) For $|z| > 1$, consider $z = 1/w$. Notice that $|w| < 1$ and
   $$1/z^2(1-z) = \frac{1}{\frac{1}{w^2}(1-\frac{1}{w})} = \frac{-w^3}{1-w}.$$
   Therefore, we have
   $$1/z^2(1-z) = -(\frac{1}{z^3})\,(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots) \quad \text{for } |z| > 1.$$
   That is, $1/z^2(1-z) = \sum_{k=3}^{\infty} -\frac{1}{z^k}$ for $|z| > 1$.

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2. Show that when $0 < |z-1| < 2$,
   $$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

   Solution. Notice that we want the Laurent series expansion centered at $z = 1$. First we do the partial fraction.
   $$\begin{aligned}\frac{z}{(z-1)(z-3)} &= \frac{(z-1)+1}{(z-1)(z-3)} \\ &= \frac{1}{z-3} + \frac{1}{(z-1)(z-3)} \\ &= \frac{1}{z-3} + \frac{1}{2}\left(\frac{1}{z-3} - \frac{1}{z-1}\right) \\ &= \frac{3}{2(z-3)} - \frac{1}{2(z-1)}\end{aligned}$$
   Moreover, $z-3 = (z-1)-2 = -2(1-\frac{z-1}{2})$, where $|\frac{z-1}{2}| < 1$ in the domain $|z-1| < 2$. Hence,
   $$\frac{1}{z-3} = -\frac{1}{2}\sum_{n=0}^{\infty}(\frac{z-1}{2})^n \quad \text{for } |z-1| < 2.$$
   The result followed by substituting back into the equation above. ◀

3. Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)^2(z+1)^2}$$

for $1 < |z| < 2$.

Solution. Let $z = 1/w$, then we have $|w| < 1$ and

$$f(z) = \frac{w^4}{(1-w)^2(1+w)^2} = \frac{w^4}{(1-w^2)^2}.$$

Recall that for any $|z| < 1$, we have

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$$

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}$$

Therefore,

$$f(z) = \frac{w^4}{(1-w^2)^2} = w^4 \sum_{k=1}^{\infty} kw^{2(k-1)} = \sum_{k=1}^{\infty} \frac{k}{z^{2k+2}}.$$

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4. Develop $\mathrm{Log}(\sin z/z)$ in powers of z up to the term z^6.

Solution. Notice that $\mathrm{Log}(\sin z/z) = \mathrm{Log}(1 - (1 - \sin z/z))$. From the power series of $\sin z$ at $z = 0$, we see that for $z \neq 0$,

$$\begin{aligned} 1 - \frac{\sin z}{z} &= 1 - \frac{1}{z}(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots) \\ &= 1 - (1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots) \\ &= \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots \end{aligned}$$

In particular, it shows that $|1 - \frac{\sin z}{z}| < 1$ when z is near 0.

Recall that

$$\mathrm{Log}(1-w) = -(w + \frac{w^2}{2} + \frac{w^3}{3} + \frac{w^4}{4} + \cdots) \quad \text{for } |w| < 1.$$

Put $w = 1 - \frac{\sin z}{z}$, we obtain

$$\begin{aligned} \mathrm{Log}(\sin z/z) &= -((1 - \frac{\sin z}{z}) + \frac{(1 - \frac{\sin z}{z})^2}{2} + \frac{(1 - \frac{\sin z}{z})^3}{3} + \cdots) \\ &= -((\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots) + \frac{1}{2}(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots)^2 + \frac{1}{3}(\frac{z^2}{3!} + \cdots)^3 + \cdots) \\ &= -((\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots) + \frac{1}{2}(\frac{z^4}{(3!)^2} - \frac{2z^6}{3!5!} + \cdots) + \frac{1}{3}(\frac{z^6}{(3!)^3} + \cdots) + \cdots) \\ &= -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2835} + \cdots \end{aligned}$$

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5. (lemma of Schwarz) Let f be an analytic function on $\{z \in \mathbb{C} : |z| < 1\}$ satisfying

   (i) $|f(z)| \le 1$,

   (ii) $f(0) = 0$,

   then we have $|f(z)| \le |z|$ and $|f'(0)| \le 1$.

   If we further assume that $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$, then $f(z) = cz$ with a constant c of absolute value 1.

Solution. Since f is analytic on $\{z \in \mathbb{C} : |z| < 1\}$, we have the Taylor series representation

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \quad \text{for } |z| < 1.$$

Assumption (ii) tells us that $a_0 = 0$. Using the Taylor series, if we put

$$f_1(z) = a_1 + a_2 z + a_3 z^2 + \cdots = \sum_{k=1}^{\infty} a_k z^{k-1},$$

then we have

$$f_1(z) = \begin{cases} \frac{f(z)}{z} & \text{if } 0 < |z| < 1; \\ a_1 & \text{if } z = 0. \end{cases}$$

The function $f_1(z)$ being a power series is also analytic on $\{z \in \mathbb{C} : |z| < 1\}$. For each $0 < r < 1$, by assumption (i), we have $|f_1(z)| \le \frac{1}{r}$ on $\{z \in \mathbb{C} : |z| = r\}$. Fix any $|z_0| < 1$, by Maximum Modulus Principle, for any r with $|z_0| < r < 1$, we have

$$|f_1(z_0)| \le \frac{1}{r}$$

Letting $r \to 1$, we see that $|f_1(z)| \le 1$ for every $|z| < 1$. Therefore, $|f(z)| \le |z|$ for $0 < |z| < 1$. On the other hand, note $a_1 = f'(0)$, by the same inequality at $z = 0$, we have $|f'(0)| = |f_1(0)| \le 1$.

In case $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$, then $|f_1(z_0)| = 1$ for some $|z_0| < 1$. Moreover, $|f_1(z_0)| \ge |f_1(z)|$ for every $|z| < 1$. By Maximum Modulus Principle, f_1 must be a constant function. That is, $f(z) = cz$ for some constant c. Check that $|c| = 1$.

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