THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5220 Complex Analysis and Its Applications 2019-20
Homework 6
Due Date: 30th April 2020

Compulsory Part

1. Use residues to evaluate the following improper integrals:

(a) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$;

(b) $\displaystyle\int_0^\infty \frac{x\sin 2x}{x^2+3}dx$.

Solution. (a) Since the integrand is an even function, we have

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx$$

We put $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ and consider the contour Γ_R composed of the upper semicircle C_R^+ centered at 0 with radius $R > 0$ and the diameter from $-R$ to R. By Cauchy's residue theorem, if $R > 2$, then

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=2i} f(z)\right) = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right) = \frac{\pi}{3}.$$

Note that

$$\left|\int_{C_R^+} f(z)\,dz\right| \le \pi R\frac{R^2}{(R^2-1)(R^2-4)} \to 0 \quad \text{as } R\to\infty.$$

Hence, we have

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{6}.$$

(b) The integrand is an even function, hence

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin 2x}{x^2+3}\,dx$$

We consider $f(z) = \frac{ze^{i2z}}{z^2+3}$ and the contour as in Q1(a). Using Cauchy's residue theorem, we have

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\sqrt{3}i} f(z) = i\pi e^{-2\sqrt{3}}.$$

On the other hand, by Jordan's lemma (because $a = 2 > 0$, see week 9 Lecture), we can conclude that

$$\int_{C_R^+} f(z)\,dz \to 0 \quad \text{as } R \to \infty.$$

Notice that

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \operatorname{Im}\left(\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz\right)$$

Therefore,

$$\int_{0}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \frac{\pi e^{-2\sqrt{3}}}{2}.$$

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2. Use residues to show that

(a) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx = -\frac{\pi}{e}\sin 2;$

(b) $\displaystyle\int_{0}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3}$, where $a > 1$.

Solution. (a) Notice that the only roots in $x^2+4x+5 = (x+2)^2+1$ are $-2+i$ and $-2-i$. Let $f(z) = \frac{e^{iz}}{z^2+4z+5}$. We consider the contour Γ_R composed of the upper semicircle centered at 0 with radius $R > 0$ and the diameter from $-R$ to R. By Cauchy residue's theorem, for large $R > 0$, we have

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-2+i} f(z) = 2\pi i\,\frac{e^{-2i-1}}{2i} = \frac{\pi}{e}\left(\cos 2 - i\sin 2\right).$$

Moreover, by Jordan's lemma (week 9 Lecture), we have

$$\int_{C_R^+} f(z)\,dz \to 0 \quad \text{as } R \to \infty.$$

Taking the imaginary part of the integral, we find that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx = -\frac{\pi}{e}\sin 2$$

(b) Since $1/(a+\cos\theta)^2$ is an even function, we have

$$\int_{0}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$$

If we put $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}(z+1/z)$ and $dz = iz d\theta$. Hence, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2} &= \int_{|z|=1} \frac{1}{\left(a+\frac{1}{2}(z+\frac{1}{z})\right)^2}\frac{dz}{iz} \\ &= \int_{|z|=1} \frac{4z^2}{\left(2az+(z^2+1)\right)^2}\frac{dz}{iz} \\ &= \int_{|z|=1} \frac{-4iz\,dz}{\left(z^2+2az+1\right)^2}\end{aligned}$$

Notice that the roots of $z^2+2az+1=(z+a)^2+1-a^2$ are $-a+\sqrt{a^2-1}$ and $-a-\sqrt{a^2-1}$. Moreover, we see that the integral is nonzero and $-a-\sqrt{a^2-1}<-1$. Hence, we can conclude that $-a+\sqrt{a^2-1}$ is the only root lying inside the circle $\{|z|=1\}$.
Let $f(z)=\frac{-4iz}{(z^2+2az+1)^2}=\frac{-4iz}{(z+a-\sqrt{a^2-1})^2(z+a+\sqrt{a^2-1})^2}$. If we put $h(z)=\frac{-4iz}{(z+a+\sqrt{a^2-1})^2}$, then

$$\operatorname*{Res}_{z=-a+\sqrt{a^2-1}} f(z)=h'(-a+\sqrt{a^2-1})=\frac{-ia}{(\sqrt{a^2-1})^3}.$$

Using residue theorem,

$$\int_{|z|=1} f(z)\,dz=2\pi i\operatorname*{Res}_{z=-a+\sqrt{a^2-1}} f(z)=\frac{2\pi a}{(\sqrt{a^2-1})^3}.$$

Therefore,

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2}=\frac{1}{2}\int_{|z|=1} f(z)\,dz=\frac{a\pi}{(\sqrt{a^2-1})^3}$$

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3. Suppose that f is analytic on and inside a positively oriented simple closed contour γ, and has no zeros on γ. If f has n zeros $z_1, z_2, \ldots, z_n$ inside γ, where z_k is of multiplicity m_k for each k, show that

$$\int_\gamma \frac{zf'(z)}{f(z)}dz=2\pi i\sum_{k=1}^n m_k z_k.$$

Solution. Let $\varphi(z)=z$. Applying the theorem on p. 7 of week 10 Lecture, we have

$$\int_\gamma \frac{zf'(z)}{f(z)}dz=2\pi i\sum_{k=1}^n m_k\varphi(z_k)=2\pi i\sum_{k=1}^n m_k z_k.$$

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4. Determine the number of zeros, counted with multiplicities, of:

(a) $z^6-6z^4+2z^3-z$ inside $|z|=1$;

(b) z^5-3z^3-z+1 inside $|z|=2$.

Solution. (a) Let $f(z)=-6z^4$ and $g(z)=z^6+2z^3-z$. Notice that

$$|g(z)|\le 1+2+1=4<6=|f(z)| \quad \text{on } |z|=1.$$

By Rouché's theorem, the functions $f(z)+g(z)=z^6-6z^4+2z^3-z$ and $f(z)$ have the same number of zeros inside $\{|z|=1\}$, which is 4.

(b) Let $f(z)=z^5$ and $g(z)=-3z^3-z+1$. Notice that

$$|g(z)|\le 3(2)^3+2+1=27<32=|f(z)| \quad \text{on } |z|=2.$$

Rouché's theorem shows that the functions $f(z)+g(z)=z^5-3z^3-z+1$ and $f(z)$ have the same number of zeros inside $\{|z|=2\}$, which is 5.

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5. Prove that $z = 1 - e^{-z}$ has exactly one solution in the right half-plane.

Solution. Fix $\epsilon > 0$. For every $R > 0$, we consider the contour composed of a line segment from $\epsilon + iR$ to $\epsilon - iR$ and the positively oriented circular arc centered at ϵ moving from $\epsilon - iR$ to $\epsilon + iR$. Notice that on the circular arc, we have

$$|1 - e^{-z}| \leq 1 + |e^{-z}| = 1 + e^{-\operatorname{Re}(z)} \leq 1 + e^{-\epsilon} \leq 2 < |z| \quad \text{if } R > 2.$$

We want to show that $|z| > |1 - e^{-z}|$ if $\operatorname{Re}(z) = \epsilon$. Let $x = \operatorname{Re}(z) = \epsilon$ and $y = \operatorname{Im}(z)$. Then, $|z|^2 = \epsilon^2 + y^2$ and $|1 - e^{-z}|^2 = (1 - e^{-\epsilon} \cos y)^2 + (e^{-\epsilon} \sin y)^2$. If we put

$$f_\epsilon(y) = |z|^2 - |1 - e^{-z}| = \epsilon^2 + y^2 - 1 + 2e^{-\epsilon} \cos y - e^{-2\epsilon},$$

then we want to show that $f_\epsilon(y) > 0$ for every $y \in \mathbb{R}$. Note

$$f'_\epsilon(y) = 2y - 2e^{-\epsilon} \sin y = 2y \left(1 - \frac{\sin y}{y}\, e^{-\epsilon}\right)$$

Therefore, $f'_\epsilon(y) > 0$ if $y > 0$ and $f'_\epsilon(y) < 0$ if $y < 0$. We have

$$f_\epsilon(y) \geq f(0) = \epsilon^2 - 1 + 2e^{-\epsilon} - e^{-2\epsilon} = \epsilon^2 - (1 - e^{-\epsilon})^2.$$

If we consider $h(x) = x - (1 - e^{-x})$ and its derivative for $x \geq 0$, we will see that $f_\epsilon(y) \geq \epsilon^2 - (1 - e^{-\epsilon})^2 > 0$ for every $y \in \mathbb{R}$.

In conclusion, $|z| > |1 - e^{-z}|$ on the whole contour. Rouché's theorem shows that $z - 1 + e^{-z}$ and z have the same number of zeros inside the contour, which is 0. Letting $R \to \infty$ and $\epsilon \to 0$, we find that $z - 1 + e^{-z}$ has no solution for $\operatorname{Re}(z) > 0$. When $\operatorname{Re}(z) = 0$, $iy = 1 - e^{-iy} = 1 - \cos y + i \sin y$. The only solution for $y = \sin y$ is $y = 0$. ◀

Optional Part

1. Use residues to evaluate the following improper integrals

   (a) $\displaystyle\int_0^\infty \frac{\cos ax}{x^2 + 4} dx;$

   (b) $\displaystyle\int_0^\pi \frac{d\theta}{5 + 4\sin\theta};$

   (c) $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2 + 1} dx;$

   (d) P.V. $\displaystyle\int_{-\infty}^\infty \frac{dx}{2x^2 + 2x + 1};$

   (e) P.V. $\displaystyle\int_{-\infty}^\infty \frac{x \sin 2x}{2x^2 + 2x + 1} dx;$

   (f) P.V. $\displaystyle\int_{-\infty}^\infty \frac{x \sin 2x}{x^2 - 1} dx.$

Solution. (a) Since cosine function is even, we may assume $a \geq 0$. Let $f(z) = \frac{e^{iaz}}{z^2+4}$. Consider the contour Γ_R composed of the upper semicircle C_R^+ of radius $R > 0$ centered at 0 and the diameter from $-R$ to R. For $a > 0$, Jordan's lemma shows that

$$\int_{C_R^+} f(z)\, dz \to 0 \quad \text{as } R \to \infty.$$

For $a = 0$, we have

$$\int_{C_R^+} f(z)\, dz \leq \pi R \frac{1}{R^2 - 4} \to 0 \quad \text{as } R \to \infty.$$

Using residue theorem, we have

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=2i} f(z) = \frac{\pi e^{-2a}}{2}$$

Taking the real part of the integral, we have

$$\int_0^\infty \frac{\cos ax}{x^2+4} dx = \frac{\pi e^{-2|a|}}{4}.$$

(b)

(c) Let $f(z) = \frac{e^{\frac{1}{2}\log z}}{z^2+1}$, where the branch of the log function is chosen to be $\frac{-3\pi}{2} < \arg z \leq \frac{\pi}{2}$. Consider the contour $\Gamma_{\epsilon,R}$ composed of two line segments and two circular arcs. The line segments are $(-R, -\epsilon)$ and (ϵ, R) and the circular arcs are upper semicircles with radii ϵ and R respectively. Using residue theorem, we have

$$\int_{\Gamma_{\epsilon,R}} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \left(\frac{e^{\frac{\pi i}{4}}}{2i} \right) = \frac{\pi}{\sqrt{2}}(1+i)$$

On the upper semicircles C_R^+ and C_ϵ^+, we have

$$\left| \int_{C_R^+} f(z)\, dz \right| \leq \frac{\pi R \sqrt{R}}{R^2 - 1}$$

$$\left| \int_{C_\epsilon^+} f(z)\, dz \right| \leq \frac{\pi \epsilon \sqrt{\epsilon}}{1 - \epsilon^2}$$

Both of them converge to 0 as $\epsilon \to 0$ and $R \to \infty$. On the line segment $(-R, -\epsilon)$, we have

$$\begin{aligned} \int_{-R}^{-\epsilon} f(z)\, dz &= \int_{-R}^{-\epsilon} \frac{e^{\frac{1}{2}(\log|z| + i\pi)}}{z^2+1}\, dz \\ &= \int_{-R}^{-\epsilon} \frac{i\sqrt{|z|}}{z^2+1}\, dz \\ &= i \int_\epsilon^R \frac{\sqrt{x}}{x^2+1}\, dx \end{aligned}$$

Letting $\epsilon \to 0$ and $R \to \infty$, we will obtain

$$\int_0^\infty \frac{\sqrt{x}}{x^2+1}\, dx = \frac{\pi}{\sqrt{2}}.$$

(d) Let $f(z) = \frac{1}{2z^2+2z+1}$. Consider the contour Γ_R composed of the upper semicircle with radius R and the diameter $(-R, R)$. Note that the roots of $2z^2 + 2z + 1 = 2(z + 1/2)^2 + 1/2$ are $\alpha = -1/2 + i/2$ and $\beta = -1/2 - i/2$. Using residue theorem, for $R > 1/2$, we have

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=\alpha} f(z) = \frac{2\pi i}{2(\alpha - \beta)} = \pi.$$

Moreover, on the upper semicircle C_R^+, we have

$$\left| \int_{C_R^+} f(z)\, dz \right| \le \frac{\pi R}{2R^2 - 2R - 1} \to 0 \quad \text{as } R \to \infty.$$

Therefore, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{2x^2 + 2x + 1} = \pi.$$

(e) Let $f(z) = \frac{ze^{i2z}}{2z^2+2z+1}$ and consider the same contour as in part (d). By Jordan's lemma, on the upper semicircle C_R^+, we have

$$\int_{C_R^+} f(z)\, dz \to 0 \quad \text{as } R \to \infty.$$

Using residue theorem, we may conclude that

$$\int_{\Gamma_R} f(z) dz = 2\pi i \operatorname*{Res}_{z=\alpha} f(z) = \frac{2\pi i \alpha e^{i2\alpha}}{2(\alpha - \beta)} = \pi \left(-\frac{1}{2} + \frac{i}{2} \right) e^{-i-1} = \frac{\pi}{\sqrt{2}e} e^{i(\frac{3\pi}{4} - 1)}$$

where α, β are those defined in part (d). Taking the imaginary part, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x}{2x^2 + 2x + 1} dx = \frac{\pi}{\sqrt{2}e} \sin\left(\frac{3\pi}{4} - 1 \right).$$

(f)

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2. Using the fact that $\sin^3 x = \text{Im}\left(\frac{3}{4}e^{ix} - \frac{1}{4}e^{i3x} - \frac{1}{2} \right)$, evaluate P.V. $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx$.

Solution. Let $f(z) = \frac{1}{z^3}\left(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2} \right)$. Consider the contour Γ_R composed of two line segments and two upper semicircles. The line segments are respectively $(-R, -\epsilon)$ and (ϵ, R) The upper semicircles are centered at 0 with radii ϵ and R. Using Cauchy's residue theorem, we have

$$\int_{\Gamma_R} f(z)\, dz = 0$$

Using Jordan's lemma and routine approximation, we can conclude that on the upper semicircle C_R^+,

$$\int_{C_R^+} f(z)\, dz \to 0 \quad \text{as } R \to \infty.$$

On the upper semicircle C_ϵ^+, we have

$$\left(\frac{3}{4}e^{iz} - \frac{1}{4}e^{i3z} - \frac{1}{2}\right) = \frac{3}{4}(1 + iz + \frac{(iz)^2}{2} + \cdots) - \frac{1}{4}(1 + i3z + \frac{(i3z)^2}{2} + \cdots) - \frac{1}{2}$$
$$= \frac{3z^2}{4} + h(z),$$

where $h(z)$ has zero of order not less than 3 at the point $z = 0$. Hence, $\int_{C_\epsilon^+} f(z)\,dz = \int_{C_\epsilon^+} \frac{3}{4z}\,dz + \int_{C_\epsilon^+} \frac{h(z)}{z^3}\,dz \to \frac{3\pi i}{4}$ as $\epsilon \to 0$. After taking the imaginary part of the integral, we have

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{4}.$$

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3. Use residues to show that

(a) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$;

(b) $\displaystyle\int_0^\infty \frac{x^a}{(x^2+1)^2} dx = \frac{(1-a)\pi}{4\cos(a\pi/2)}$, where $-1 < a < 3$.

Solution. (a) Let $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2}$ and consider the contour composed of the upper semicircle of radius R and the diameter $(-R, R)$. It is routine to show that the integral over the upper semicircle goes to 0 as $R \to \infty$. Now, it suffices to calculate $\operatorname{Res}_{z=3i} f(z)$ and $\operatorname{Res}_{z=2i} f(z)$. Note that

$$\operatorname*{Res}_{z=3i} f(z) = \frac{(3i)^2}{6i((3i)^2+4)^2} = \frac{-3}{50i}.$$

On the other hand, to find $\operatorname{Res}_{z=2i} f(z)$, we observe that it is the coefficient of $(z - 2i)$ in the Taylor series expansion of $\frac{z^2}{(z^2+9)(z+2i)}$ at $z = 2i$. Moreover,

$$\frac{z^2}{(z^2+9)(z+2i)} = \left(1 - \frac{9}{z^2+9}\right)\frac{1}{(z+2i)^2} = f_1(z)f_2(z).$$

The required residue is $f_1(2i)f_2'(2i) + f_1'(2i)f_2(2i) = \frac{13}{200i}$.
Using residue theorem, we can conclude that

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx = \pi i(\operatorname*{Res}_{z=3i} f(z) + \operatorname*{Res}_{z=2i} f(z)) = \frac{\pi}{200}.$$

(b) Let $f(z) = \frac{e^{a\log z}}{(z^2+1)^2}$, where the branch is taken to be $-\frac{\pi}{2} < \arg z \le \frac{3\pi}{2}$. Consider the contour composed of two line segments and two upper semicircles. The line segments are $(-R, -\epsilon)$ and (ϵ, R). The upper semicircles are centered at 0 with radii ϵ and R respectively. It is routine to check that the integrals of $f(z)$ over these two semicircles would go to 0, as $\epsilon \to 0$ and $R \to \infty$. On the other hand,

$$\begin{aligned}\int_{-R}^{-\epsilon} f(z)\, dz &= \int_{-R}^{-\epsilon} \frac{e^{a(\log|z|+i\pi)}}{(z^2+1)^2}\, dz \\ &= \int_{\epsilon}^{R} \frac{x^a e^{ia\pi}}{(x^2+1)^2}\, dx\end{aligned}$$

Finally, we would calculate $\mathrm{Res}_{z=i} f(z)$. Note that the required residue is the coefficient of $(z-i)$ in the Talyor series expansion of $e^{a\log z}/(z+i)^2$ at the point $z=i$. If we put $f_1(z) = e^{a\log z}$ and $f_2(z) = \frac{1}{(z+i)^2}$, then the coefficent is given by

$$\begin{aligned}f_1(i)f_2'(i) + f_1'(i)f_2(i) &= e^{\frac{i\pi a}{2}}\left(\frac{-2}{(2i)^3}\right) + \frac{a}{i}e^{\frac{i\pi a}{2}}\left(\frac{1}{(2i)^2}\right) \\ &= \frac{1-a}{4i}e^{\frac{i\pi a}{2}}.\end{aligned}$$

Therefore, letting $\epsilon \to 0$ and $R \to \infty$, together with residue theorem, we have

$$(1+e^{ia\pi})\int_0^\infty f(z)\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = \frac{(1-a)\pi}{2}e^{\frac{i\pi a}{2}}.$$

After dividing $(1+e^{ia\pi})$ on both sides, we will obtain the desired result.

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4. Use Rouché's theorem to show that all the zeros of $z^5 + 3z^2 + 7$ are contained inside the open disk $|z| < 2$.

Solution. Let $f(z) = z^5$ and $g(z) = 3z^2 + 7$. Notice that on the circle $\{|z| = 2\}$, we have

$$|g(z)| \le 3(2)^2 + 7 = 19 < 32 = |f(z)|.$$

Rouché's theorem tells us that the function $f(z) + g(z) = z^5 + 3z^2 + 7$ have the same number of zeros as $f(z)$ inside the circle, which is 5. Since it is just a degree 5 polynomial, all zeros are contained inside the open disk $|z| < 2$. ◀