Strominger-Yau-Zaslow Transformations in Mirror Symmetry

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Abstract of thesis entitled:

Strominger-Yau-Zaslow Transformations in Mirror Symmetry

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We study mirror symmetry via Fourier-Mukai-type transformations, which we call SYZ mirror transformations, in view of the ground-breaking Strominger-Yau-Zaslow Mirror Conjecture which asserted that the mirror symmetry for Calabi-Yau manifolds could be understood geometrically as a T-duality modified by suitable quantum corrections. We apply these transformations to investigate a case of mirror symmetry with quantum corrections, namely the mirror symmetry between the A-model of a toric Fano manifold $\bar{X}$ and the B-model of a Landau-Ginzburg model (Y, W). Here Y is a noncompact Kähler manifold and $W : Y \to \mathbb{C}$ is a holomorphic function. We construct an explicit SYZ mirror transformation which realizes canonically the isomorphism

$$QH^*(\bar{X}) \cong Jac(W)$$

between the quantum cohomology ring of $\bar{X}$ and the Jacobian ring of the function W. We also show that the symplectic structure $\omega_{\bar{X}}$ of $\bar{X}$ is transformed to the holomorphic volume form $e^W \Omega_Y$ of (Y, W). Concerning the Homological Mirror Symmetry Conjecture, we exhibit certain correspondences between A-branes on $\bar{X}$ and B-branes on (Y, W) by applying the SYZ philosophy.

論文摘要

我們通過 Fourier-Mukai-型變換研究鏡對稱。有鑑於獨創性的 Strominger-Yau-Zaslow 鏡猜想，我們稱這些為 SYZ 變換；此猜想斷言 Calabi-Yau 流型的鏡對稱可於幾何上理解為利用適當的量子校正修定過的T-對偶。我們利用這些 SYZ 變換去考察一個有量子校正的鏡對稱，即一個環 Fano 流型 $\bar{X}$ 的 A-模型與一個 Landau-Ginzburg 模型 (Y, W) 的 B-模型之間的鏡對稱；這裡 Y 是一個非緊 Kähler 流型，而 $W : Y \to \mathbb{C}$ 是一個全純函數。我們構造一個明確的 SYZ 變換去標準地實踐 $\bar{X}$ 的量子上同調環與函數 W 的 Jacobian 環的同構

$$QH^*(\bar{X}) \cong Jac(W).$$

同時我們顯示 $\bar{X}$ 的辛結構 $\omega_{\bar{X}}$ 被轉化為 (Y, W) 的全純體積型式。關於同調鏡對稱猜想，我們應用 SYZ 原理來展示某些 $\bar{X}$ 上的 A-膜與 (Y, W) 上的 B-膜的對應。

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To my wife, Dawn

Contents

Chapter 1

Introduction

1.1 What is mirror symmetry?

Mirror symmetry is a phenomenal consequence of *superstring theory* [22]. The latter is an attempt to unify the two pillars of twentieth century theoretical physics: general relativity and quantum mechanics. Consistency of superstring theory requires spacetime to have ten dimensions, four of which is the classical spacetime, i.e. three spatial dimensions plus one time dimension; the remaining six curl up into a very tiny gadget, known as a *Calabi-Yau threefold* in geometry. While there are plenty examples of such spaces [11], the universe should be modeled on only one of them. In the search of this candidate, physicists discovered that superstring theories (or more precisely, superconformal field theories) on two entirely different Calabi-Yau threefolds[1] can occasionally be equivalent [23]. This duality is not only surprising from a physical point of view, but also leads to many astonishing and highly nontrivial mathematical consequences.

One of these being the famous prediction of the numbers of rational curves on a quintic threefold [10]. The story goes as

[1]In most cases, even the topology of the Calabi-Yau threefolds are different.

follows. Consider the hypersurface X in $\mathbb{C}P^4$ defined by

$$X = \{z \in \mathbb{C}P^4 : f(z) = 0\},$$

where f is a generic homogeneous polynomial of degree five. X is a Calabi-Yau threefold by a simple calculation of Chern classes. A long standing question in algebraic geometry is the computation of the number $N_X(d)$ of rational curves of degree d in X for any $d \in H_2(X, \mathbb{Z}) \cong \mathbb{Z}$. At the time of the discovery of mirror symmetry, these numbers were virtually incalculable by mathematical methods and geometers could only strive to get them one by one. But by exploiting mirror symmetry, physicists were able to give a complete answer for all $N_X(d)$ at once using much easier calculations. More precisely, the computation of the numbers $N_X(d)$ was reduced to a calculation involving the *variations of Hodge structures* on another Calabi-Yau threefold Y, called the *mirror manifold* of X[2]. The latter question is well-studied and is related to the complex algebraic geometry of Y; while the number of rational curves turn out to be symplectic invariants (Gromov-Witten invariants) of X. Indeed, at least morally, mirror symmetry for Calabi-Yau manifolds can be formulated as follows.

For any Calabi-Yau manifold X, there exists another Calabi-Yau manifold Y, called the mirror manifold of X, such that the symplectic geometry (A-model) of X is equivalent to the complex algebraic geometry (B-model) of Y, and vice versa.

Besides quintic threefolds, many other examples exhibiting similar phenomena have been found. However, it is much more desirable to know why mirror symmetry works. Two attempts were made towards a mathematical understanding of mirror

[2]This is called the mirror theorem for quintic threefolds, which was proved by Givental [21] and Lian-Liu-Yau [34] independently. For details, see the book of Cox-Katz [14]

symmetry for Calabi-Yau manifolds. In an address to the 1994 International Congress of Mathematicians in Zürich, Kontsevich [29] speculated that mirror symmetry for a pair of Calabi-Yau manifolds X and Y could be explained as a statement in homological algebra. His *Homological Mirror Symmetry Conjecture* says the following.

There is an equivalence of triangulated categories

$$D^b Fuk(X) \cong D^b Coh(Y)$$

where $D^b Fuk(X)$ is the derived category of the Fukaya A_∞-category $Fuk(X)$ of X and $D^b Coh(Y)$ is the derived category of coherent sheaves of Y.

In contrast to this algebraic formulation of Kontsevich, the ground-breaking work of Strominger-Yau-Zaslow [43] suggested a geometric explanation of mirror symmetry. The *Strominger-Yau-Zaslow Mirror Conjecture* asserts that

Any Calabi-Yau manifold X admits a fibration by special Lagrangian tori and the mirror Calabi-Yau manifold Y can be obtained by T-duality, i.e. dualizing the special Lagrangian torus fibration of X. Moreover, the A-model of X should be interchanged with the B-model of Y, and vice versa, through Fourier-type transformations.

Those Fourier-type transformations are going to play a key role in this thesis and will be called *SYZ mirror transformations*. Both the Homological Mirror Symmetry Conjecture and the SYZ Mirror Conjecture are based on considerations of physical objects called *D-branes* in superstring theory. These two conjectures are expected to reveal the secret of mirror symmetry for Calabi-Yau manifolds. Unfortunately, only in a few examples (namely elliptic curves and quartic K3 surfaces) have mathematicians been able to verify the Homological Mirror Symmetry Conjecture; and even worse, not a single nontrivial example has been found for the SYZ Mirror Conjecture (see, however, the recent breakthrough by Gross-Siebert [24], after earlier works of Fukaya [15] and Kontsevich and Soibelman [31]).

On the other hand, mirror symmetry was extended to other settings, notably to Fano manifolds [20], [30], [28], [26]. Given a Fano manifold $\bar{X}$, its mirror is conjecturally given by a noncompact Kähler manifold Y together with a holomorphic function $W : Y \to \mathbb{C}$. The pair (Y, W) is called a *Landau-Ginzburg model* in physics, and the function W is called the *superpotential*. The A-model (respectively, B-model) of the Landau-Ginzburg model (Y, W) refers to the symplectic-(respectively, complex-) geometric aspects of the singularity theory of the holomorphic function W on Y. Again, mirror symmetry predicts that the A- and B-models of $\bar{X}$ and (Y, W) are interchanged, and this has many mathematical consequences. An example is the following conjecture:

There exists a ring isomorphism between the quantum cohomology $QH^*(\bar{X})$ of a Fano manifold $\bar{X}$ and the Jacobian ring $Jac(W)$ of the superpotential W of the mirror Landau-Ginzburg model (Y, W).

This conjecture has been verified (at least) for toric Fano and flag manifolds by the works of Givental [17], [18] and many others. A version of the Homological Mirror Symmetry Conjecture has also been formulated by Kontsevich [30]. In this case, the conjecture must be stated as two separate halves since the categories associated to the Landau-Ginzburg model (Y, W) have to be suitably modified.

There are equivalences of triangulated categories

$$\begin{aligned} D^bCoh(\bar{X}) &\cong D^bFuk(Y,W), and \\ D^bFuk(\bar{X}) &\cong D_{Sing}(Y,W), \end{aligned}$$

where $Fuk(Y,W)$ is a variant of the usual Fukaya category [40] and $D_{Sing}(Y,W)$ is the category of singularities of (Y,W) [39].

Substantial evidences in the toric case [40], [44], [6], [7], [1], [2], [13], [12] have been found (in contrast to the non-toric and Calabi-Yau cases where evidence is much rarer). Nevertheless, a geometric explanation for the mirror symmetry phenomenon for Fano manifolds, in particular, an analogue of the SYZ Mirror Conjecture, is lacking.

1.2 Statements of main results

The aim of this thesis is to explore the geometry of mirror symmetry for toric Fano manifolds following the SYZ philosophy. In particular, we will interpret the mirror symmetry for toric Fano manifolds as a T-duality modified by suitable quantum corrections and establish a canonical correspondence between the A-model of a toric Fano manifold and the B-model of the mirror Landau-Ginzburg model by means of SYZ mirror transformations.

To describe our results, we first fix some notations. Let $N \cong \mathbb{Z}^n$ be a lattice and $M = \mathrm{Hom}(N,\mathbb{Z})$ the dual lattice. Let $\bar{X}$ be an n-dimensional toric Fano manifold associated to a fan in $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. If $v_1, \ldots, v_d \in N$ are the primitive generators of the one-dimensional cones of the fan defining $\bar{X}$, then a polytope $\bar{P} \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ defined by the inequalities

$$\langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d,$$

for some $\lambda_1, \ldots, \lambda_d \in \mathbb{R}$ associates a symplectic or Kähler structure $\omega_{\bar{X}}$ to $\bar{X}$. Physicists predicted that the mirror of $(\bar{X},$

$\omega_{\bar{X}}$), as a symplectic manifold, is given by the Landau-Ginzburg model (Y, W), where Y is the non-compact Kähler manifold $(\mathbb{C}^*)^n$ and $W : Y \to \mathbb{C}$ is the Laurent polynomial

$$e^{\lambda_1} z^{v_1} + \ldots + e^{\lambda_d} z^{v_d}$$

where z^{v_i} denotes the monomial $z_1^{v_i^1} \ldots z_n^{v_i^n}$ and $z_1, \ldots, z_n$ are complex coordinates on Y.

We need torus fibrations in order to apply T-duality. One of the advantages of $\bar{X}$ being a toric manifold is that the moment map $\mu : \bar{X} \to \bar{P}$ of the Hamiltonian T^n-action on $\bar{X}$ provides a natural Lagrangian torus fibration. Moreover, the restriction to the open dense orbit $X \subset \bar{X}$ is a torus bundle

$$\mu : X \to P,$$

where $P = \bar{P} \backslash \partial \bar{P}$ is the interior of $\bar{P}$[3]. Our first result says that the mirror manifold Y, derived by Hori-Vafa [28] using physical arguments, can essentially be obtained by a direct application of T-duality to $\mu : X \to P$. More precisely, we will prove the following proposition in Chapter 2, Section 2.3.

Proposition 1.2.1. *Let Y_{SYZ} be the total space of the dual torus fibration of $\mu : X \to P$. Then Y_{SYZ}, which we call the semi-flat SYZ mirror of X, is the open complex submanifold*

$$\{(z_1, \ldots, z_n) \in Y : |e^{\lambda_i} z^{v_i}| < 1 \text{ for } i = 1, \ldots, d\}$$

contained in Hori-Vafa's mirror manifold $Y = (\mathbb{C}^*)^n$.

Much more mysterious is the superpotential $W : Y \to \mathbb{C}$. Intuitively, W is "mirror" to the toric divisor at infinity $D_\infty = \bigcup_{i=1}^d D_i = \bar{X} \setminus X$. Here D_i denotes the toric prime divisor which corresponds to $v_i \in N$ for $i = 1, \ldots, d$. Note that the moment map $\mu : \bar{X} \to \bar{P}$ is singular exactly along D_∞ and all the

[3]According to the definition of Auroux [5], the fibers of $\mu : X \to P$ are special Lagrangians.

quantum corrections, namely holomorphic curves and discs, are due to the compactification of $X \cong (\mathbb{C}^*)^n$ by adding D_∞. Since we have ignored the toric divisor at infinity $D_\infty = \bigcup_{i=1}^d D_i = \bar{X} \setminus X$ and hence quantum corrections, we are unable to see the superpotential W just by using T-duality. To recapture the information, we consider the cover

$$\pi : \tilde{X} := X \times N \to X$$

and various functions on it. Let $\mathcal{K} \subset H^2(\bar{X}, \mathbb{R})$ be the *Kähler cone* of $\bar{X}$. For each $q = (q_1, \ldots, q_l) \in \mathcal{K}$ (where $l := d - n =$ Picard number of $\bar{X}$), we introduce a function $\Phi_q \in C^\infty(\tilde{X})$ as *a generating function for holomorphic discs counting on* $\bar{X}$. This gives a family of functions $\{\Phi_q\} \subset C^\infty(\tilde{X})$ over $\mathcal{K}$. Now, if we assume that $\bar{X}$ is a product of projective spaces, then $\{\Phi_q\}$ can be used to compute the quantum cohomology $QH^*(\bar{X})$ and the quantum product naturally becomes a convolution product (see Chapter 3, Section 3.1).

Proposition 1.2.2.

1. *The logarithmic derivatives of* Φ_q*, with respect to* q_a *for* $a = 1, \ldots, l$*, are given by*

$$q_a \frac{\partial \Phi_q}{\partial q_a} = \Phi_q \star \Psi_{n+a}.$$

Here $\Psi_i \in C^\infty(\tilde{X})$ *is defined, for* $i = 1, \ldots, d$*, by Maslov index two holomorphic discs which correspond to the toric divisor* D_i*, and* $\star$ *denotes the convolution product with respect to the lattice* N*.*

2. *We have a natural isomorphism of* $\mathbb{C}$*-algebras*

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}$$

where $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ *is the polynomial algebra generated by* $\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm}$ *with respect to the convolution product* $\star$*, and* $\mathcal{L}$ *is the ideal generated by linear relations that are defined by the linear equivalence among the toric divisors* $D_1, \ldots, D_d$*, provided that* $\bar{X}$ *is a product of projective spaces.*

A key to the proof is the observation that each holomorphic curve which contributes to the quantum product can be obtained by gluing of Maslov index two holomorphic discs. This can be seen explicitly and made concrete if we go to the world of tropical geometry [36]. We have to use the fundamental results of Cho and Oh [13] on the classification of holomorphic discs in toric Fano manifolds. The condition that X is a product of projective spaces is imposed since we want to give a geometric proof of the proposition, without appealing to Givental's mirror theorem [18].

Now the upshot is that we can explicitly construct and apply SYZ mirror transformations to study the mirror symmetry between $\bar{X}$ and (Y, W). More precisely, the *SYZ mirror transformation for toric Fano manifolds* $\mathcal{F}$ will be defined to be *a combination of the semi-flat SYZ transformation and taking fibrewise Fourier series.* We prove that the SYZ transformation of Φ_q is precisely the exponential of the superpotential W, i.e. $\mathcal{F}(\Phi_q) = e^W$, and show how the symplectic structure of $\bar{X}$ is transformed to the holomorphic structure of (Y, W) (see Chapter 3, Section 3.2).

Theorem 1.2.1. *The SYZ transformation of the generating function* Φ_q *for holomorphic discs counting on the toric Fano manifold* $\bar{X}$ *gives (the exponential of) the superpotential* W *on the mirror manifold* $Y \cong (\mathbb{C}^*)^n$*:*

$$\mathcal{F}(\Phi_q) = e^W \in H^0(Y, \mathcal{O}_Y).$$

Furthermore, we can incorporate the symplectic structure $\omega_{\bar{X}}$ *on* $\bar{X}$ *to give the holomorphic structure on the Landau-Ginzburg* *model (Y, W) through SYZ transformation in the sense that*

$$\mathcal{F}(\Phi_q e^{\omega_{\bar{X}}}) = e^W \Omega_Y.$$

Here we view $\Phi_q e^{\omega_{\bar{X}}}$ as a *symplectic structure corrected by holomorphic discs* and $e^W \Omega_Y$ as a *holomorphic volume form.*

As another application of our approach, we give a geometric explanation of the isomorphism $QH^*(\bar{X}) \cong Jac(W)$. In more details, We show that the SYZ transformation $\mathcal{F}(\Psi_i)$ of the function Ψ_i is nothing but the monomial $e^{\lambda_i} z^{v_i}$, for $i = 1, \ldots, d$, and thereby exhibiting a canonical isomorphism between the quantum cohomology $QH^*(\bar{X})$ and the Jacobian ring $Jac(W)$, which takes the quantum product $*$, now realized as the convolution product $\star$, to the ordinary product of functions, just as what classical theory of Fourier series does (see Chapter 3, Section 3.3).

Theorem 1.2.2. *The SYZ transformation induces a canonical isomorphism of $\mathbb{C}$-algebras*

$$\mathcal{F} : QH^*(\bar{X}) \xrightarrow{\cong} Jac(W)$$

provided that $\bar{X}$ is a product of projective spaces.

Regarding the Homological Mirror Symmetry Conjecture, we will consider the simplest correspondence between D-branes, namely the correspondence between an A-brane $(L, \mathbb{L})$, where L is a Lagrangian torus fiber of X and $\mathbb{L}$ is a flat $U(1)$-bundle over L, and the mirror B-brane $(z, \mathcal{O}_z)$, where $z \in Y$ is a point and $\mathcal{O}_z$ is the skyscraper sheaf. By using Cho and Oh's results [13], [12], we show in Chapter 3, Section 3.4 that

Proposition 1.2.3. *The endomorphism algebra of the A-brane $(L, \mathbb{L})$, which is given by the Floer cohomology $HF(L, \mathbb{L})$, is isomorphic to the endomorphism algebra $End(z, \mathcal{O}_z)$ of the mirror B-brane $(z, \mathcal{O}_z)$ as $\mathbb{C}$-algebras. In particular, the Floer cohomology $HF(L, \mathbb{L})$ of $(L, \mathbb{L})$ is nontrivial if and only if the mirror $z \in Y$ is a critical point of $W : Y \to \mathbb{C}$.*

The rest of this thesis is organized as follows. In Chapter 2, we briefly review the use of SYZ transformations in mirror symmetry without quantum corrections, namely mirror symmetry for semi-flat Calabi-Yau manifolds, and give a proof of Proposition 1.2.1. Chapter 3 is the heart of this thesis, where we construct the SYZ mirror transformation for toric Fano manifolds explicitly and show how they can be applied to give a geometric understanding of mirror symmetry for toric Fano manifolds. In particular, we will prove Theorems 1.2.1 and 1.2.2 and Proposition 1.2.3, and provide a couple of examples for illustration. In the final section, we conclude with discussions of possible future research directions.

□ **End of chapter.**

Chapter 2

Mirror symmetry without corrections

In this chapter, following Leung [32], we will briefly review mirror symmetry for *semi-flat Calabi-Yau manifolds*, where quantum corrections are absent. We will see that mirror symmetry in this case is T-duality without any modifications. As an application, we prove Proposition 1.2.1.

2.1 Review of mirror symmetry for semi-flat Calabi-Yau manifolds

As before, $N \cong \mathbb{Z}^n$ and $M = \mathrm{Hom}(N, \mathbb{Z})$ denote dual lattices and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ denote respectively the real vector spaces spanned by N, M. We also denote by T_N (and T_M) the real n-torus $N_{\mathbb{R}}/N$ (and $M_{\mathbb{R}}/M$). Let $D \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ be a convex domain. (More generally, instead of a convex domain, one may consider an affine manifold.) Out of D, we can naturally construct two manifolds which are mirror to each other.

First of all, the tangent bundle $TD = D \times iM_{\mathbb{R}}$ is naturally a complex manifold with complex coordinates $z_j = x_j + iy_j$ where x_j's and y_j's are respectively the base coordinates on D and fiber coordinates on $M_{\mathbb{R}}$. We have the standard holomorphic volume form $\Omega_{TD} = dz_1 \wedge \ldots \wedge dz_n$ on TD. By taking quotient by the lattice $M \subset M_{\mathbb{R}}$, we can compactify the fiber directions to give the complex manifold

$$Y = TD/M = D \times iT_M,$$

where T_M denotes the torus $M_{\mathbb{R}}/M$. It is naturally equipped with a torus fibration (in fact a torus bundle)

$$\nu_Y : Y \to D.$$

The holomorphic n-form Ω_{TD} descends to give the holomorphic volume form

$$\Omega_Y = dz_1 \wedge \ldots \wedge dz_n$$

on Y. Moreover, if ϕ is an elliptic solution of the *real Monge-Ampère equation*

$$\det\Big(\frac{\partial^2 \phi}{\partial x_j \partial x_k}\Big) = const,$$

then the Kähler form

$$\omega_Y = i\partial\bar{\partial}\phi = \sum_{j,k} \phi_{jk} dx_j \wedge dy_k,$$

with ϕ_{jk} denoting $\frac{\partial^2 \phi}{\partial x_j \partial x_k}$, determines a Calabi-Yau metric on Y, and $\nu_Y : Y \to D$ becomes a special Lagrangian torus fibration, which is called an *SYZ fibration*. In summary, we have the following structures on Y:

Riemannian metric	$g_Y = \sum_{j,k} \phi_{jk}(dx_j \otimes dx_k + dy_j \otimes dy_k)$
Holomorphic volume form	$\Omega_Y = \bigwedge_{j=1}^n (dx_j + idy_j)$
Symplectic form	$\omega_Y = \sum_{j,k} \phi_{jk} dx_j \wedge dy_k$
SYZ fibration	$\nu_Y : Y \to D$

The SYZ Mirror Conjecture [43] suggested that the mirror manifold X of Y should be given by the moduli space of pairs (L, ∇) where L is a special Lagrangian torus fiber of Y and ∇ is a flat $U(1)$-connection on the trivial line bundle $L \times \mathbb{C}$ over L. This is nothing but the total space of the dual torus fibration $\mu_X : X = D \times iT_N \to D$, taking into account the fact that the dual torus $T_M = (T_N)^*$ is the moduli space of flat $U(1)$-connections on the trivial line bundle $T_N \times \mathbb{C}$ over the torus T_N.

Furthermore, X can naturally be viewed as the quotient of the cotangent bundle $T^*D = D \times iN_{\mathbb{R}}$ by the lattice $N \subset N_{\mathbb{R}}$. In particular, the standard symplectic form $\omega_{T^*D} = \sum_{j=1}^n dx_j \wedge du_j$, where u_j's are the fiber coordinates on $N_{\mathbb{R}}$, descends to give a symplectic form

$$\omega_X = \sum_{j=1}^n dx_j \wedge du_j$$

on $X = T^*D/N$. Through the metric

$$g_X = \sum_{j,k} (\phi_{jk} dx_j \otimes dx_k + \phi^{jk} du_j \otimes du_k),$$

where (ϕ^{jk}) is the inverse matrix of (ϕ_{jk}), we obtain a complex structure on X with complex coordinates given by $dw_j = \sum_{k=1}^n \phi_{jk} dx_k + i du_j$. There is a corresponding holomorphic volume form which can be written as

$$\Omega_X = dw_1 \wedge \ldots \wedge dw_n = \bigwedge_{j=1}^n (\sum_{k=1}^n \phi_{jk} dx_k + i du_j).$$

Also the projection map

$$\mu_X : X \to D$$

naturally becomes a special Lagrangian torus fibration. In summary, we have the following structures on X:

Riemannian metric	$g_X = \sum_{j,k}(\phi_{jk}dx_j \otimes dx_k + \phi^{jk}du_j \otimes du_k)$
Holomorphic volume form	$\Omega_X = \bigwedge_{j=1}^n(\sum_{k=1}^n \phi_{jk}dx_k + idu_j)$
Symplectic form	$\omega_X = \sum_{j=1}^n dx_j \wedge du_j$
SYZ fibration	$\mu_X : X \to D$

It is easy to see that the geometric structures of X and Y are interchanged by T-duality or dualizing torus fibrations. Hence we call X the *semi-flat SYZ mirror* of Y (and vice versa).

We remark that both Y and X admit natural Hamiltonian T^n-actions, but while $\mu : X \to D$ is a moment map for the T_N-action on X, $\nu : Y \to D$ is not a moment map for the T_M-action on Y. In fact, a moment map $\mu_Y : Y \to N_{\mathbb{R}}$ for the T_M-action on Y is given by

$$\mu_Y = L_\phi \circ \nu_Y,$$

where $L_\phi : D \to N_{\mathbb{R}}$ is the Legendre transform of ϕ defined by

$$L_\phi(x_1, \ldots, x_n) = d\phi_x = \Big(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}\Big).$$

Since ϕ is convex, the image $D^* = L_\phi(D)$ is an open convex subset of $(M_{\mathbb{R}})^* = N_{\mathbb{R}}$. (For this and other properties of the Legendre transform, see, for example, the book of Guillemin [25], Appendix 1.) In the action coordinates $x^1, \ldots, x^n$ of D^*, which are given by $\frac{\partial x^j}{\partial x_k} = \phi_{jk}$, the various structures on Y are written as

Riemannian metric	$g_Y = \sum_{j,k}(\phi^{jk}dx^j \otimes dx^k + \phi_{jk}dy_j \otimes dy_k)$
Holomorphic volume form	$\Omega_Y = \bigwedge_{j=1}^n(\sum_{k=1}^n \phi^{jk}dx^k + idy_j)$
Symplectic form	$\omega_Y = \sum_{j=1}^n dx^j \wedge dy_j$
SYZ fibration	$\mu_Y : Y \to D^*$

We move on to discuss the correspondences between D-branes in this case. Lying at the heart of the SYZ Mirror Conjecture is the basic but important observation that a point $z = x + iy \in Y$ defines a flat $U(1)$-connection ∇_y on the trivial line bundle over the special Lagrangian torus fiber $F_x := \mu_X^{-1}(x)$. Now the point $z \in Y$ together with its structure sheaf $\mathcal{O}_z$ constitute a *B-brane* on Y; while the pair $(L_x, \mathbb{L}_y)$, where $\mathbb{L}_y$ denotes the flat $U(1)$-bundle $(L_x \times \mathbb{C}, \nabla_y)$, gives an *A-brane* on X. This implements the simplest case of correspondence between D-branes on mirror manifolds:

$$(L_x, \mathbb{L}_y) \longleftrightarrow (z, \mathcal{O}_z).$$

The space of infinitestimal deformations of the A-cycle $(L_x, \mathbb{L}_y)$, which is given by $H^1(L_x, \mathbb{C})$, is canonically identified with the tangent space T_zY under T-duality.

On the other hand, consider a section $L = \{(x, u(x)) \in X : x \in D\}$ of $\mu_X : X \to D$. The submanifold L is Lagrangian if and only if (locally) there exists a function f such that $u_j = \frac{\partial f}{\partial x_j}$. By the above observation (now used in the opposite way), a point $(x, u(x)) \in L$ determines a flat $U(1)$-connection $\nabla_{u(x)}$ on the trivial bundle over the fiber $\Gamma_x = \nu_Y^{-1}(x)$. Therefore, the family of points $\{(x, u(x)) : x \in D\}$ gives rise to the $U(1)$-connection

$$\nabla_L = d_Y - \frac{i}{2}\sum_{j=1}^{n} u_j(x) dy_j$$

on the trivial bundle over Y. Its curvature two form is given by

$$F_L = d_Y\Big(-\frac{i}{2}\sum_{j=1}^{n} u_j(x) dy_j\Big) = -\frac{i}{2}\sum_{j,k} \frac{\partial u_j}{\partial x_k} dx_k \wedge dy_j,$$

and, in particular,

$$F_L^{2,0} = \frac{1}{8}\sum_{j<k} \Big(\frac{\partial u_j}{\partial x_k} - \frac{\partial u_k}{\partial x_j}\Big) dz_j \wedge dz_k.$$

We conclude that ∇_L is integrable, i.e. $F_L^{2,0} = 0$, if and only if L is Lagrangian. More generally, we can equip L with a flat $U(1)$-bundle $\mathbb{L} = (L \times \mathbb{C}, d_L + \alpha)$, where $\alpha \in \Omega^1(L, \mathbb{R})$ is a closed (and hence exact) one-form. The A-brane $(L, \mathbb{L})$ is then transformed to the $U(1)$-connection

$$\nabla_{L,\mathbb{L}} = \nabla_L + \alpha,$$

which again is integrable if and only if L is Lagrangian. Furthermore, one can prove that $\nabla_{L,\mathbb{L}}$ satisfies the *deformed Hermitian-Yang-Mills equations* if and only if L is special Lagrangian (see [33] and [32] for the detailed proofs). $\nabla_{L,\mathbb{L}}$ is a connection on a line bundle over Y given by the semi-flat SYZ transformation of $\mathbb{L}$:

$$\mathcal{L}_{L,\mathbb{L}} = \pi_{Y,*}(\pi_X^*(\iota_*\mathbb{L}) \otimes \mathcal{P}),$$

where $\iota : L \hookrightarrow X$ is the inclusion map. In conclusion, the A-brane $(L, \mathbb{L})$ corresponds to the B-brane $(Y, \mathcal{L}_{L,\mathbb{L}})$ again through T-duality:

$$(L, \mathbb{L}) \longleftrightarrow (Y, \mathcal{L}_{L,\mathbb{L}}).$$

2.2 Semi-flat SYZ transformations

In this section, we will see how the geometric structures of the mirror manifolds X and Y are transformed to each other by fiberwise Fourier-type transformations.

We have mentioned that the dual torus $(T_N)^* = T_M$ can be interpreted as the moduli space of flat $U(1)$-bundles on T_N. More precisely, given $y = (y_1, \ldots, y_n) \in M_\mathbb{R}$, we have a connection

$$\nabla_y = d + \frac{i}{2}\sum_{j=1}^{n} y_j du_j$$

on the topologically trivial line bundle $T_N \times \mathbb{C}$ over T_N. This is a flat $U(1)$-connection and it is gauge equivalent to the trivial connection if and only if $y \in M$. Moreover this construction gives all flat $U(1)$-connections on $T_N \times \mathbb{C}$ up to unitary gauge transformations. The universal $U(1)$-bundle is given by the topologically trivial line bundle $\mathcal{P} = T_N \times T_M \times \mathbb{C}$ (the so-called *Poincaré bundle*) over the product $T_N \times T_M$, equipped with the universal connection $d + \frac{i}{2}\sum_{j=1}^{n}(y_j du_j - u_j dy_j)$. The curvature of this connection is the universal two form

$$F = i\sum_{j=1}^{n} dy_j \wedge du_j.$$

Now consider the relative version of this picture. Let $X \times_D Y = D \times i(T_N \times T_M)$ be the fiber product of the fibrations $\mu : X \to D$ and $\nu : Y \to D$. By abuse of notations, we still use $\mathcal{P}$ and $F = i\sum_{j=1}^{n} dy_j \wedge du_j \in \Omega^2(X \times_D Y)$ to denote the fiberwise universal line bundle and curvature two form respectively.

Definition 2.2.1. *The semi-flat SYZ mirror transformation $\mathcal{F} : \Omega^*(X) \to \Omega^*(Y)$ is defined by*

$$\mathcal{F}(\alpha) = \pi_{Y,*}(\pi_X^*(\alpha) \wedge e^F) = \int_{T_N} \pi_X^*(\alpha) \wedge e^F$$

where $\pi_X : X \times_D Y \to X$ and $\pi_Y : X \times_D Y \to Y$ are the two natural projections.

The point is that this fiberwise Fourier-type transformation transforms the symplectic structure on X to the complex structure on Y in the sense of the following

Proposition 2.2.1.

$$\mathcal{F}(e^{\omega_X}) = \Omega_Y.$$

Proof. The proof is by direct calculation.

$$\begin{aligned}
\mathcal{F}(e^{\omega_X}) &= \int_{T_N} \pi_X^*(e^{\omega_X}) \wedge e^F \\
&= \int_{T_N} e^{\sum_{j=1}^n (dx_j + i dy_j)\wedge du_j} \\
&= \int_{T_N} \bigwedge_{j=1}^n \big(1 + (dx_j + i dy_j) \wedge du_j\big) \\
&= \int_{T_N} \left(\bigwedge_{j=1}^n (dx_j + i dy_j) \right) \wedge du_1 \wedge \ldots \wedge du_n \\
&= \Omega_Y
\end{aligned}$$

where we have assumed that $\int_{T_N} du_1 \wedge \ldots \wedge du_n = 1$. $\square$

Moreover, $\mathcal{F}$ has the inversion property, which demonstrates a property which a mirror transformation should possess:

Proposition 2.2.2. *If we define the inverse transform* $\mathcal{F}^{-1} : \Omega^*(Y) \to \Omega^*(X)$ *by*

$$\mathcal{F}^{-1}(\alpha) = i^{-n} \pi_{X,*}(\pi_Y^*(\alpha) \wedge e^{-F}) = i^{-n} \int_{T_M} \pi_Y^*(\alpha) \wedge e^{-F},$$

then

$$\mathcal{F}^{-1}(\Omega_Y) = e^{\omega_X}.$$

Proof.

$$\begin{aligned}
\mathcal{F}^{-1}(\Omega_Y) &= i^{-n}\int_{T_M}\pi_Y^*(\Omega_Y)\wedge e^{-F}\\
&= i^{-n}\int_{T_M}\left(\bigwedge_{j=1}^{n}(dx_j+idy_j)\right)\wedge e^{i\sum_{j=1}^{n}du_j\wedge dy_j}\\
&= i^{-n}\int_{T_M}\bigwedge_{j=1}^{n}\big((dx_j+idy_j)\wedge e^{idu_j\wedge dy_j}\big)\\
&= i^{-n}\int_{T_M}\bigwedge_{j=1}^{n}\big(dx_j+idy_j+idx_j\wedge du_j\wedge dy_j\big)\\
&= i^{-n}\int_{T_M}\bigwedge_{j=1}^{n}\big(ie^{dx_j\wedge du_j}\wedge dy_j\big)\\
&= \int_{T_M} e^{\sum_{j=1}^{n}dx_j\wedge du_j}\wedge dy_1\wedge\ldots\wedge dy_n\\
&= e^{\omega_X},
\end{aligned}$$

where we have again assumed that $\int_{T_M} dy_1\wedge\ldots\wedge dy_n=1$. □

By exactly the same arguments, one can also show that

$$\mathcal{F}(\Omega_X)=e^{\omega_Y},\ \mathcal{F}^{-1}(e^{\omega_Y})=\Omega_X.$$

If we include the consideration of B-fields, the semi-flat SYZ transformation will give an identification of the moduli space of complexified symplectic structures on X with the moduli space of complex structures on Y, and vice versa. For a more detailed discussion of this and other things, we refer to Leung [32].

2.3 Derivation of Hori-Vafa's mirror manifold

Recall that, for a toric Fano manifold $\bar{X}$, the primitive generators of the 1-dimensional cones of the fan defining $\bar{X}$ are denoted by $v_1, \ldots, v_d$. Without loss of generality, we can assume that $v_1 = e_1, \ldots, v_n = e_n$ is the standard basis of $N \cong \mathbb{Z}^n$. The map

$$\partial : \mathbb{Z}^d \to N, \ (k_1, \ldots, k_d) \mapsto \sum_{i=1}^{d} k_i v_i$$

is surjective since $\bar{X}$ is compact. Let K be the kernel of ∂, so that the sequence

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\partial} N \longrightarrow 0 \tag{2.3.1}$$

is exact. Now, if

$$Q_1 = (Q_{11}, \ldots, Q_{d1}), \ldots, Q_l = (Q_{1l}, \ldots, Q_{dl}) \in \mathbb{Z}^d$$

is a $\mathbb{Z}$-basis of K, then the mirror manifold of $\bar{X}$, derived by Hori and Vafa in [28] using physical arguments, is the complex submanifold

$$Y = \Big\{(Z_1, \ldots, Z_d) \in (\mathbb{C}^*)^d : \sum_{i=1}^{d} Q_{ia} Z_i = r_a, \quad a = 1, \ldots, l\Big\},$$

in $(\mathbb{C}^*)^d$, where $r_a = -\sum_{i=1}^{d} Q_{ia}\lambda_i$ for $a = 1, \ldots, l$. In terms of these coordinates, the superpotential $W : Y \to \mathbb{C}$ is given by

$$W = e^{-Z_1} + \ldots + e^{-Z_d}.$$

The goal of this section is to show that Hori-Vafa's mirror manifold Y naturally arises as we apply T-duality to the open dense orbit $X \subset \bar{X}$. To do this, we have to recall the construction of $\bar{X}$ (and X) as a *symplectic quotient*.

From the above exact sequence (2.3.1), we get an exact sequence of real tori

$$0 \longrightarrow T_K \xrightarrow{\iota} T^d \xrightarrow{\partial} T_N \longrightarrow 0 \tag{2.3.2}$$

where $T_K := K_\mathbb{R}/K$. Considering their Lie algebras and dualizing the sequence gives

$$0 \longrightarrow M_\mathbb{R} \xrightarrow{\partial^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} K_\mathbb{R}^* \longrightarrow 0.$$

The standard diagonal action of T^d on $\mathbb{C}^d$ is Hamiltonian with respect to the standard symplectic form $\frac{i}{2}\sum_{i=1}^d dW_i \wedge d\bar{W}_i$ and the moment map $h : \mathbb{C}^d \to (\mathbb{R}^d)^*$ is given by

$$h(W_1, \ldots, W_d) = \frac{1}{2}(|W_1|^2, \ldots, |W_d|^2).$$

Restricting to T_K, we get a Hamiltonian action of T_K on $\mathbb{C}^d$ with moment map $h_K = \iota^* \circ h$. Using the $\mathbb{Z}$-basis $\{Q_1, \ldots, Q_l\}$ of K, the map $\iota^* : (\mathbb{R}^d)^* \to K_\mathbb{R}^*$ is given by

$$\iota^*(X_1, \ldots, X_d) = \left(\sum_{i=1}^d Q_{i1} X_i, \ldots, \sum_{i=l}^d Q_{il} X_i \right) \tag{2.3.3}$$

in the coordinates associated to the dual basis $Q_1^*, \ldots, Q_l^*$ of K^*. The moment map $h_K : \mathbb{C}^d \to K_\mathbb{R}^*$ can thus be written as

$$h_K(W_1, \ldots, W_d) = \frac{1}{2}\left(\sum_{i=1}^d Q_{i1}|W_i|^2, \ldots, \sum_{i=1}^d Q_{il}|W_i|^2 \right) \in \mathbb{R}^l \cong K_\mathbb{R}^*.$$

Note that $r := (r_1, \ldots, r_l)$ lies in $K_\mathbb{R}^*$. Now $\bar{X}$ and X are given by the symplectic quotients

$$\bar{X} = h_K^{-1}(r)/T_K,\ X = (h_K^{-1}(r) \cap (\mathbb{C}^*)^d)/T_K$$

respectively (see, for example, Guillemin [25], appendix).

In this process, notice that the image of $h_K^{-1}(r)$ under the map $h : \mathbb{C}^d \to (\mathbb{R}^d)^*$ lies inside the affine linear subspace $M_\mathbb{R}(r) = \{(X_1, \ldots, X_d) \in (\mathbb{R}^d)^* : \iota^*(X_1, \ldots, X_d) = r\}$ which is a translate of $M_\mathbb{R}$. Hence, restricting h to $h_K^{-1}(r) \cap (\mathbb{C}^*)^d$ gives a T^d-invariant bundle $h : h_K^{-1}(r) \cap (\mathbb{C}^*)^d \to P \subset M_\mathbb{R}(r)$. Taking quotient by T_K will give a T_N-bundle

$$\mu : X \to P.$$

In fact, we can write (see [3])

$$X = T^*P/N = P \times iT_N$$

and the induced symplectic form $\omega_X = \omega_{\bar{X}}|_X$ is the standard symplectic form:

$$\omega_X = \sum_{j=1}^{n} dx_j \wedge du_j$$

where x_j's and u_j's (with $u_j \equiv u_j + 2\pi$) are respectively the coordinates on $P \subset M_{\mathbb{R}}(r)$ and T_N. In other words, the x_j's and u_j's are symplectic or *action-angle coordinates.* We are therefore in exactly the same situation as in Section 2.1.

Applying T-duality, i.e. dualizing the torus bundle $\mu : X \to P$, we get the semi-flat SYZ mirror manifold $Y_{SYZ} = TP/M = P \times iT_M$, together with the standard complex structure and holomorphic n-form:

$$\Omega_{Y_{SYZ}} = \bigwedge_{j=1}^{n} (dx_j + idy_j) = dz_1 \wedge \ldots \wedge dz_n$$

where y_j's (with $y_j \equiv j_j + 2\pi$) are the dual coordinates on $T_M = (T_N)^*$ and $z_j = x_j + iy_j$ are the complex coordinates on Y_{SYZ}. We also have the dual torus fibration

$$\nu : Y_{SYZ} \to P.$$

We are now in a position to prove Proposition 1.2.1.

Proof of Proposition 1.2.1. Dualizing the sequence (2.3.2) and complexifying gives

$$0 \longrightarrow T_M^{\mathbb{C}} \xrightarrow{\partial^*} (\mathbb{C}^*)^d \xrightarrow{\iota^*} (T_K^*)^{\mathbb{C}} \longrightarrow 0.$$

If $Z_i = X_i + iY_i$, $i = 1, \ldots, d$ are the complex coordinates on $(\mathbb{C}^*)^d$, then it follows from the definition of the map $\iota^* : (\mathbb{R}^d)^* \to$

$K^*_{\mathbb{R}}$ given in (2.3.3) that Hori-Vafa's mirror manifold Y can be written as

$$Y = \Big\{(Z_1, \dots, Z_d) \in (\mathbb{C}^*)^d : \iota^*(Z_1, \dots, Z_d) = r\Big\}.$$

From this we see that Y is just the complex submanifold $M_{\mathbb{R}}(r) \times iT_M \subset (\mathbb{C}^*)^d$ and the inclusion map $\partial^* : Y_{HV} \hookrightarrow (\mathbb{C}^*)^d$ is given, in the complex coordinates $z_j = x_j + iy_j$ on $Y = M_{\mathbb{R}}(r) \times iT_M \cong (\mathbb{C}^*)^n$, by

$$\partial^*(z_1, \dots, z_n) = (\langle z, v_1 \rangle - \lambda_1, \dots, \langle z, v_d \rangle - \lambda_d).$$

Since $\sum_{i=1}^d Q_{ia} v_i = 0$ for $a = 1, \dots, l$, we have

$$\sum_{i=1}^d Q_{ia}(\langle z, v_i \rangle - \lambda_i) = r_a, \quad a = 1, \dots, l$$

and it follows that $Z_i|_Y = \langle z, v_i \rangle - \lambda_i$ for $i = 1, \dots, d$. Now Y_{SYZ} is given by the open subset

$$Y_{SYZ} = \{(z_1, \dots, z_n) \in Y : \mathrm{Re}(\langle z, v_i \rangle - \lambda_i) > 0 \text{ for } i = 1, \dots, d\}.$$

$\square$

By the proposition, the SYZ mirror Y_{SYZ} is strictly smaller than Hori-Vafa's mirror Y. This issue was discussed in Hori-Vafa [28] and Auroux [5] and we refer to those papers for details. From now on, we would be confusing the notations and using Y to denote either the SYZ semi-flat mirror or Hori-Vafa's mirror manifold. Which one we are referring to should be clear from the context.

$\square$ **End of chapter.**

Chapter 3

Mirror symmetry for toric Fano manifolds

In the previous chapter, we have shown that it is enough to obtain the mirror manifold Y of a toric Fano manifold $\bar{X}$ using T-duality by just considering the Lagrangian torus bundle $\mu : X \to P$ given by the restriction of the moment map $\mu : \bar{X} \to \bar{P}$ to the open dense orbit $X \subset \bar{X}$. But as we mentioned in the introduction, to get the superpotential $W : Y \to \mathbb{C}$, we must retain the information of the compactification of $X \cong (\mathbb{C}^*)^n$ by adding the toric divisor D_∞ at infinity. To do this, recall that we consider the cover

$$\pi : \tilde{X} := X \times N \to X.$$

In this chapter, we will see how $\tilde{X}$ and functions on it help to recover the quantum information and enable us to define the SYZ mirror transformations for the toric Fano manifold $\bar{X}$ and get the superpotential W.

3.1 A generating function of holomorphic discs and $QH^*(\bar{X})$

In this section, we introduce a generating function Φ_q of holomorphic discs and show its relation to the quantum cohomology of $\bar{X}$. In particular, we demonstrate that the quantum cup product can be realized as a convolution product (Proposition 1.2.2).

Recall that from Section 2.3, we have the exact sequence (2.3.1):

$$0 \longrightarrow K \xrightarrow{\iota} \mathbb{Z}^d \xrightarrow{\partial} N \longrightarrow 0. \tag{3.1.1}$$

Now consider the Kähler cone $\mathcal{K} \subset H^2(\bar{X}, \mathbb{R})$ of $\bar{X}$. For each $q = (q_1, \dots, q_l) \in \mathcal{K}$ ($l = d-n$), we let $\bar{P}$ be the polytope defined by

$$\bar{P} = \{x \in M_{\mathbb{R}} : \langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \dots, d\}$$

with $\lambda_i = 0$ for $i = 1, \dots, n$ and $\lambda_{n+a} = \log q_a$ for $a = 1, \dots, l$, where as before, we assume that $v_1 = e_1, \dots, v_n = e_n$ is the standard basis of $N \cong \mathbb{Z}^n$. For a point $x \in P = \bar{P} \setminus \partial \bar{P}$, we let $L_x = \mu^{-1}(x)$ be the Lagrangian torus fibre over x. Then the groups $H_2(\bar{X}, \mathbb{Z})$, $\pi_2(\bar{X}, L_x)$ and $\pi_1(L_x)$ can be identified canonically with K, $\mathbb{Z}^d$ and N respectively, so that the above exact sequence (3.1.1) coincides with the following exact sequence of homotopy groups associated to the pair $(\bar{X}, L_x)$:

$$0 \longrightarrow H_2(\bar{X}, \mathbb{Z}) \xrightarrow{\iota} \pi_2(\bar{X}, L_x) \xrightarrow{\partial} \pi_1(L_x) \longrightarrow 0.$$

It is known that $\pi_2(\bar{X}, L_x)$ is generated by d classes $\beta_1, \dots, \beta_d$. By the results of Cho-Oh [13], there is a unique (up to automorphism of the domain) holomorphic disc $\varphi_i : (D^2, \partial D^2) \to (\bar{X}, L_x)$ representing the class β_i. The symplectic area of φ_i is given by (Cho-Oh [13], Theorem 8.1):

$$\int_{\beta_i} \omega_{\bar{X}} = \int_{D^2} \varphi_i^* \omega_{\bar{X}} = \langle x, v_i \rangle - \lambda_i. \tag{3.1.2}$$

Definition 3.1.1. *Let*

$$\pi_2^+(\bar{X}, L_x) = \left\{ \sum_{i=1}^{d} k_i \beta_i \in \pi_2(\bar{X}, L_x) : k_i \in \mathbb{Z}_{\geq 0}, \ i = 1, \dots, d \right\}$$

be the positive cone generated by $\beta_1, \ldots, \beta_d$. *For* $\beta = \sum_{i=1}^d k_i\beta_i \in \pi_2^+(\bar{X}, L_x)$, *we let* $w(\beta) = \prod_{i=1}^d k_i!$ *be its weight. For* $q = (q_1, \ldots, q_l) \in \mathcal{K}$, *define the function* $\Phi_q \in C^\infty(P \times N)$ *by*

$$\Phi_q(x, v) = \sum_{\beta \in \pi_2^+(\bar{X}, L_x),\ \partial\beta = v} \frac{1}{w(\beta)} e^{-\int_\beta \omega_{\bar{X}}}.$$

Remark 3.1.1.

1. *Let* $\bar{P}$ *be the polytope defined by the inequalities*

   $$\langle x, v_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d.$$

   Also let

   $$Q_1 = (Q_{11}, \ldots, Q_{d1}), \ldots, Q_l = (Q_{1l}, \ldots, Q_{dl}) \in \mathbb{Z}^d$$

   be a $\mathbb{Z}$*-basis of* K. *Then the point* $q = (q_1, \ldots, q_l) \in \mathcal{K}$ *in the Kähler cone is given by* $q_a = e^{-r_a}$ *for* $a = 1, \ldots, l$, *where*

   $$r_a = -\sum_{i=1}^d Q_{ia}\lambda_i.$$

   Hence, choosing a polytope which gives rise to the Kähler structure parametrized by $q \in \mathcal{K}$ *is equivalent to choosing a* $\mathbb{Z}$*-basis of the lattice* $K = H_2(\bar{X}, \mathbb{Z})$. *Our choice of the polytope* $\bar{P}$ *amounts to choosing the* $\mathbb{Z}$*-basis* $\{Q_1, \ldots, Q_l\}$ *of* K *such that* $(Q_{n+a,b})_{1 \leq a,b \leq l} = Id_l$.

2. *If we consider suitably marked holomorphic discs, then the weight* $w(\beta)$ *of the class* β *should account for the redundance in counting disconnected holomorphic discs. This deserves further clarification.*

We are going to prove the first part of Proposition 1.2.2. But the reader may have noticed that for different $q \in \mathcal{K}$, the function $\Phi_q \in C^\infty(P \times N)$ is defined on a different space, as P depends on q. What we do would be using analytic continuation and regarding all Φ_q as functions defining on $M_{\mathbb{R}} \times N$. For functions $f, g \in C^\infty(M_{\mathbb{R}} \times N)$, we define their convolution $f \star g \in C^\infty(M_{\mathbb{R}} \times N)$ by

$$f \star g(x, v) = \sum_{v_1 + v_2 = v} f(x, v_1) g(x, v_2).$$

Now we are ready to state and prove:

Proposition 3.1.1 (=part 1. of Proposition 1.2.2)**.** *The logarithmic derivatives of Φ_q, with respect to q_a for $a = 1, \ldots, l$, are given by*

$$q_a \frac{\partial \Phi_q}{\partial q_a} = \Phi_q \star \Psi_{n+a}$$

where $\Psi_i \in C^\infty(M_{\mathbb{R}} \times N)$ is defined, for $i = 1, \ldots, d$, by

$$\Psi_i(x, v) = \begin{cases} e^{-\int_{\beta_i} \omega_{\bar{X}}} & \text{if } v = v_i \\ 0 & \text{if } v \neq v_i. \end{cases}$$

Here, again we use analytic continuation so that Ψ_i is defined on $M_{\mathbb{R}} \times N$.

Proof. For simplicity, we will just compute $q_l \frac{\partial \Phi_q}{\partial q_l}$. By using Cho-Oh's formula (3.1.2) and our choice of the polytope $\bar{P}$, we have

$$e^{\langle x, v \rangle} \Phi_q(x, v) = \sum_{k_1, \ldots, k_d \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^d k_i v_i = v} \frac{q_1^{k_{n+1}} \ldots q_l^{k_d}}{k_1! \ldots k_d!}.$$

Note that the right-hand-side is independent of $x \in P$. Differentiating both sides with respect to q_l gives

$$\begin{aligned} e^{\langle x,v\rangle}\frac{\partial \Phi_q}{\partial q_l}(x,v) &= \sum_{k_1,\ldots,k_{d-1}\in\mathbb{Z}_{\geq 0},\ k_d\in\mathbb{Z}_{\geq 1},\ \sum_{i=1}^d k_i v_i = v} \frac{q_1^{k_{n+1}}\ldots q_{l-1}^{k_{d-1}} q_l^{k_d-1}}{k_1!\ldots k_{d-1}!(k_d-1)!} \\ &= \sum_{k_1,\ldots,k_d\in\mathbb{Z}_{\geq 0},\ \sum_{i=1}^d k_i v_i = v-v_d} \frac{q_1^{k_{n+1}}\ldots q_l^{k_d}}{k_1!\ldots k_d!} \\ &= e^{\langle x,v-v_d\rangle}\Phi_q(x,v-v_d). \end{aligned}$$

Hence, we obtain

$$q_l\frac{\partial \Phi_q}{\partial q_l} = q_l e^{-\langle x,v_d\rangle}\Phi_q(x,v-v_d).$$

Now, by definition of convolution, we have

$$\Phi_q \star \Psi_d(x,v) = \sum_{v_1+v_2=v}\Phi_q(x,v_1)\Psi_d(x,v_2) = \Phi_q(x,v-v_d)\Psi_d(x,v_d),$$

and $\Psi_d(x,v_d) = e^{-\int_{\beta_d}\omega_X} = e^{\lambda_d-\langle x,v_d\rangle} = q_l e^{-\langle x,v_d\rangle}$. The result follows. $\square$

In the previous proposition, we introduce functions $\Psi_i \in C^\infty(M_\mathbb{R}\times N)$ for $i=1,\ldots,d$. Each is defined by the unique holomorphic disc $\varphi_i : (D^2,\partial D^2) \to (\bar{X}, L_x)$ representing the class β_i, and by the classification results of Cho and Oh, these are all the Maslov index two discs with boundary lying on the torus Lagrangian fibre L_x. It is also easy to see that, for each $i=1,\ldots,d$, φ_i is the unique Maslov index two holomorphic disc which intersects D_i at one interior point (which can be chosen to be the center of the disc). We therefore conclude that the functions $\{\Psi_i\}$, the holomorphic discs $\{\varphi_i\}$ and the toric divisors $\{D_i\}$ are in one-to-one correspondences with each other:

$$\{\Psi_i\} \stackrel{1-1}{\longleftrightarrow} \{\varphi_i\} \stackrel{1-1}{\longleftrightarrow} \{D_i\}.$$

Each toric divisor D_i determines a cohomology class in $H^2(\bar{X}, \mathbb{C})$ which will again be denoted by D_i. By the above one-to-one correspondences, we can introduce linear relations in the $\mathbb{C}$-vector space spanned by the functions $\{\Psi_i\}$ using the linear equivalences among the divisors $\{D_i\}$.

Definition 3.1.2. *Two linear combinations $\sum_{i=1}^d a_i\Psi_i$ and $\sum_{i=1}^d b_i\Psi_i$ are said to be linearly equivalent, denoted by $\sum_{i=1}^d a_i\Psi_i \sim \sum_{i=1}^d b_i\Psi_i$, if the corresponding divisors $\sum_{i=1}^d a_iD_i$ and $\sum_{i=1}^d b_iD_i$ are linearly equivalent.*

We further define Ψ_i^{-1}, $i = 1, \ldots, d$, by

$$\Psi_i^{-1}(x, v) = \begin{cases} e^{\int_{\beta_i}\omega_X} & \text{if } v = -v_i \\ 0 & \text{if } v \neq v_i \end{cases}$$

so that $\Psi_i^{-1} \star \Psi_i = 1$. Now recall the second part of Proposition 1.2.2.

Proposition 3.1.2 (=part 2. of Proposition 1.2.2)**.** *We have a natural isomorphism of $\mathbb{C}$-algebras*

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}$$

where $\mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$ is the polynomial algebra generated by $\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}$ with respect to the convolution product and $\mathcal{L}$ is the ideal generated by linear equivalences, provided that $\bar{X}$ is a product of projective spaces.

It is known by the general theory of toric varieties (c.f. [16], [4]) that the cohomology ring $H^*(\bar{X}, \mathbb{C})$ of a compact toric manifold $\bar{X}$ is generated by the classes $D_1, \ldots, D_d$ in $H^2(\bar{X}, \mathbb{C})$. More precisely, there is a presentation of the form:

$$H^*(\bar{X}, \mathbb{C}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR})$$

where $\mathcal{L}$ is the ideal generated by linear equivalences and $\mathcal{SR}$ is the *Stanley-Reisner ideal* generated by *primitive relations* (see

Batyrev [8]). Now by a result of Siebert and Tian [42], $QH^*(\bar{X})$ is also generated by $D_1, \ldots, D_d$ and a presentation of $QH^*(\bar{X})$ can be given by replacing each relation in $\mathcal{SR}$ by the quantum counterpart. Denote by $\mathcal{SR}_Q$ the quantum counterpart of the Stanley-Reisner ideal. Then we can rephrase what we said as:

$$QH^*(\bar{X}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q).$$

The computation of $QH^*(\bar{X})$ (as a presentation) therefore reduces to computing the generators of the ideal $\mathcal{SR}_Q$.

Let $\bar{X} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_l}$ be a product of projective spaces. The dimension of $\bar{X}$ is $n := n_1 + \ldots + n_l$. For $a = 1, \ldots, l$, denote by $v_{1,a} = e_1, \ldots, v_{n_a,a} = e_{n_a}, v_{n_a+1,a} = -\sum_{j=1}^{n_a} e_j \in N_a$ the primitive generators of the 1-dimensional cones in the fan of $\mathbb{P}^{n_a}$, where $\{e_1, \ldots, e_{n_a}\}$ is the standard basis of $N_a \cong \mathbb{Z}^{n_a}$. We use the same symbol $v_{j,a}$ to denote the vector

$$(0, \ldots, \underbrace{v_{j,a}}_{a\text{-th}}, \ldots, 0) \in N = N_1 \oplus \ldots \oplus N_l$$

where $v_{j,a}$ sits in the a-th place, for $j = 1, \ldots, n_a+1$, $a = 1, \ldots, l$. These $d = \sum_{a=1}^{l}(n_a+1) = n+l$ vectors in N generate the fan of X. We also denote by $D_{j,a}$ the toric divisor, $\varphi_{j,a} : (D^2, \partial D^2) \to (\bar{X}, L_x)$ the holomorphic disc, $\beta_{j,a} \in \pi_2(\bar{X}, L_x)$ the homotopy class and $\Psi_{j,a} \in C^\infty(M_{\mathbb{R}} \times N)$ the function corresponding to $v_{j,a}$. Using these notations, all the primitive relations can be explicitly written down:

Lemma 3.1.1. *There are exactly l primitive collections:*

$$\mathfrak{P}_a = \{v_{j,a} : j = 1, \ldots, n_a + 1\},$$

$a = 1, \ldots, l$, *and hence*

$$\mathcal{SR} = \langle \prod_{j=1}^{n_a+1} D_{j,a} : a = 1 \ldots, l \rangle.$$

Proof. Let $\mathfrak{P}$ be any primitive collection. Suppose $\mathfrak{P} \not\subset \mathfrak{P}_a$ for any a. Then for each a such that $\mathfrak{P} \cap \mathfrak{P}_a \neq \emptyset$, we can find $v \in \mathfrak{P} \setminus (\mathfrak{P} \cap \mathfrak{P}_a)$. By definition, $\mathfrak{P} \setminus \{v\}$ generates a cone in the fan of $\bar{X}$. But all the cones in the fan of $\bar{X}$ are direct sums of cones in the fans of the factors. So in particular, $\mathfrak{P} \cap \mathfrak{P}_a$ will generate a cone in the fan of $\mathbb{P}^{n_a}$. However, this will imply that the set $\mathfrak{P}$ itself generates a cone, which is impossible. We conclude that $\mathfrak{P}$ must be contained in, and hence equal to one of the $\mathfrak{P}_a$'s. The result follows. $\square$

As a result, the cohomology ring of $\bar{X} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_l}$ is given by

$$H^*(\bar{X}, \mathbb{C}) = \frac{\mathbb{C}[D_{1,1}, \ldots, D_{n_l+1,l}]}{\left\langle D_{j,a} - D_{n_a+1,a} : j = 1, \ldots, n_a,\ a = 1, \ldots, l \right\rangle + \left\langle \prod_{j=1}^{n_a+1} D_{j,a} : a = 1, \ldots, l \right\rangle}.$$

Recall that $\pi_2^+(\bar{X}, L_x)$ is the positive cone generated by the classes $\beta_{1,1}, \ldots, \beta_{n_l+1,l}$. Then by a theorem of Batyrev [8], the kernel of $\partial : \pi_2^+(\bar{X}, L_x) \to \pi_1(L_x)$ is the *effective cone* $H_2^{\text{eff}}(\bar{X}, \mathbb{Z})$, i.e. the cone of classes which are represented by holomorphic curves. For $a = 1, \ldots, l$, denote by δ_a the effective class $\sum_{j=1}^{n_a+1} \beta_{j,a} \in H_2^{\text{eff}}(\bar{X}, \mathbb{Z})$. Then $H_2^{\text{eff}}(\bar{X}, \mathbb{Z}) = \langle \delta_a : a = 1, \ldots, l \rangle$ and the Kähler parameters $q = (q_1, \ldots, q_a)$ are given by

$$q_a = e^{-\int_{\delta_a} \omega_{\bar{X}}}.$$

We can now compute $\mathcal{SR}_Q$ by the following lemma:

Lemma 3.1.2. *Fix a point $p \in X$ so that $\mu(p) = x$. Then there is a unique (up to automorphism of the domain) holomorphic curve $\varphi_a : (\mathbb{P}^1; x_0, x_1, \ldots, x_{n_a+1}) \to \bar{X}$ with $n_a + 2$ marked points representing the class $\delta_a \in H_2(\bar{X}, \mathbb{Z})$ such that $\varphi_a(x_j) \in D_{j,a}$ for $j = 1, \ldots, n_a + 1$ and $\varphi_a(x_0) = p$. Moreover, the Gromov-Witten invariant*

$$GW_{0,n_a+2,\gamma}(P.D.(pt); D_{1,a}, \ldots, D_{n_a+1,a}, T_i) = \begin{cases} 1 & \text{if } \gamma = \delta_a \text{ and } T_i = [pt] \\ 0 & \text{otherwise} \end{cases}$$

where the T_i's is a basis of $H^*(\bar{X};\mathbb{C})$.

Proof. For the proof of the first part, see Batyrev [9], Theorem 9.3 (see also [41], section 4). For the second part, observe that since $\bar{X}$ is convex, the Gromov-Witten invariants are enumerative, in the sense that only smooth irreducible curves will contribute. Also, any $\gamma \in H_2^{\text{eff}}(\bar{X},\mathbb{Z})$ can be written as $\gamma = \sum_{a=1}^{l}\sum_{r=1}^{n_a+1} a_{r,a}v_{r,a}$ where all $a_{r,a} \geq 0$. Let $\varphi : (\mathbb{P}^1; x_0, x_1, \ldots, x_{n_a+1}) \to \bar{X}$ be a curve representing γ such that $\varphi(x_j) \in D_{j,a}$ for $j = 1, \ldots, n_a+1$ and $\varphi(x_0) = p$. Then $a_{j,a} \geq 1$ for $j = 1, \ldots, n_a+1$, and so $\sum_{a=1}^{l}\sum_{r=1}^{n_a+1} a_{r,a} \geq n_a + 1$. But if the Gromov-Witten invariant $\mathrm{GW}_{0,n_a+2,\gamma}(\text{P.D.}(\text{pt}); D_{1,a}, \ldots, D_{n_a+1,a}, T)$ is not zero, then by dimension counting, we must have

$$\begin{aligned}2((n_a+2-3)+(n_a+1))+\deg(T)\\ = 2(\dim(\bar{X})-3+(n_a+2)+\sum_{a=1}^{l}\sum_{r=1}^{n_a+1} a_{r,a})\end{aligned}$$

whence $\deg(T)+2(n_a+1) = 2\dim(\bar{X})+2\sum_{a=1}^{l}\sum_{r=1}^{n_a+1} a_{r,a}$. This is possible only if $\deg(T) = 2\dim(\bar{X})$ and $\gamma = \delta_a = \sum_{j=1}^{n_a+1} v_{j,a}$. Now the second part follows from the first part. □

By the second part of the above lemma, we have the relation

$$D_{1,a} * \ldots * D_{n_a+1,a} = e^{-\int_{\delta_a}\omega_{\bar{X}}} = q_a$$

in $QH^*(\bar{X})$. Hence

$$\mathcal{SR}_Q = \langle \prod_{j=1}^{n_a+1} D_{j,a} - q_a : a = 1\ldots, l\rangle.$$

We are in a position to prove Proposition 3.1.2:

Proof of Proposition 3.1.2. For a general toric Fano manifold $\bar{X}$, notice that we have the relation

$$\begin{aligned}
&\Psi_1^{Q_{1a}} \star \ldots \star \Psi_n^{Q_{na}} \star \Psi_{n+a}(x,v) \\
= &\begin{cases} e^{-\sum_{i=1}^{n} Q_{ia}\int_{\beta_i}\omega_{\bar{X}} - \int_{\beta_{n+a}}\omega_{\bar{X}}} & \text{if } v = 0 \\ 0 & \text{otherwise} \end{cases} \\
= &\begin{cases} q_a & \text{if } v = 0 \\ 0 & \text{otherwise} \end{cases} \\
= &\; q_a,
\end{aligned}$$

or $\Psi_{n+a} = q_a(\Psi^{-1})^{Q_{1a}} \star \ldots \star (\Psi_n^{-1})^{Q_{na}}$. For each $i = 1, \ldots, n$, there exists $1 \leq a \leq l$ such that $Q_{ia} > 0$ (since otherwise all v_i will lie in some half space of $N_{\mathbb{R}}$ which is impossible). Thus the inclusion

$$\mathbb{C}[\Psi_1, \ldots, \Psi_n, \Psi_{n+1}, \ldots, \Psi_d] \hookrightarrow \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]$$

is an isomorphism. Consider the surjective map

$$\rho : \mathbb{C}[D_1, \ldots, D_d] \to \mathbb{C}[\Psi_1, \ldots, \Psi_d]$$

defined by mapping D_i to Ψ_i for $i = 1, \ldots, d$. This map is not injective because there are nontrivial relations in $\mathbb{C}[\Psi_1, \ldots, \Psi_d]$ which are generated by the relations

$$\Psi_1^{Q_{1a}} \star \ldots \star \Psi_n^{Q_{na}} \Psi_{n+a} - q_a = 0, \; a = 1, \ldots, l.$$

By the above lemma, in the case of products of projective spaces, the kernel of ρ is exactly given by the ideal $\mathcal{SR}_Q$, so that we have an isomorphism

$$\mathbb{C}[D_1, \ldots, D_d]/\mathcal{SR}_Q \xrightarrow{\cong} \mathbb{C}[\Psi_1, \ldots, \Psi_d].$$

Since $(\mathbb{C}[D_1, \ldots, D_d]/\mathcal{SR}_Q)/\mathcal{L} = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q) = QH^*(\bar{X})$, we have the desired isomorphism

$$QH^*(\bar{X}) \cong \mathbb{C}[\Psi_1, \ldots, \Psi_d]/\mathcal{L} \cong \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L}.$$

□

Remark 3.1.2. *By Givental's mirror theorem [18], Proposition 3.1.2 is true for all toric Fano manifolds $\bar{X}$.*

The geometry of the isomorphism in Proposition 3.1.2 can be explained as follows (see the examples in Section 3.3). The quantum corrections brought by the relation

$$D_{1,a} * \ldots * D_{n_a+1,a} = q_a$$

is due to the holomorphic curve $\varphi_a : (\mathbb{P}^1; x_0, x_1, \ldots, x_{n_a+1}) \to \bar{X}$ which passes through the divisors $D_{1,a}, \ldots, D_{n_a+1,a}$ and the point $p \in X$ above $x \in P$. On the other hand, the holomorphic discs $\varphi_{1,a}, \ldots, \varphi_{n_a+1,a}$ are responsible for the relation

$$\Psi_{1,a} \star \ldots \star \Psi_{n_a+1,a} = q_a.$$

In fact, the holomorphic curve φ_a can be obtained by gluing together the holomorphic discs $\varphi_{1,a}, \ldots, \varphi_{n_a+1,a}$. We can make this clearer by going to the world of *tropical geometry*. We first consider a diffeomorphism $P \to N_{\mathbb{R}}$ given by the Legendre transform (see [3])

$$x \mapsto \left(\frac{\partial g_P}{\partial x_1}, \ldots, \frac{\partial g_P}{\partial x_n} \right)$$

of the strictly convex function $g_P : P \to \mathbb{R}$ defined by

$$g_P(x) = \sum_{a=1}^{l} \sum_{j=1}^{n_a+1} (\lambda_{j,a} - \langle x, v_{j,a} \rangle) \log(\langle x, v_{j,a} \rangle - \lambda_{j,a}).$$

We need this diffeomorphism because tropical curves and discs will lie in $N_{\mathbb{R}}$, instead of P (for details of tropical geometry, please refer to [35], [36], [38]).

Proposition 3.1.3. *The set-theoretic union of the tropical discs associated to the holomorphic discs $\varphi_{1,a}, \ldots, \varphi_{n_a+1,a}$ is a tropical curve $\Gamma \subset N_{\mathbb{R}}$ with only 1 vertex at $g_P(x) \in N_{\mathbb{R}}$ and $n_a + 1$ unbounded edges in the directions $v_{1,a}, \ldots, v_{n_a+1,a}$.*

Proof. The tropical disc associated to $\varphi_{j,a}$ is a half line emanating from the given point $g_P(x) \in N_{\mathbb{R}}$ with slope $v_{j,a}$ (see [13], [37]). The union of these half lines is a graph with only 1 vertex at $g_P(x)$ of valence $n_a + 1$. The proposition follows from the fact that the balancing condition

$$\sum_{j=1}^{n_a+1} v_{j,a} = 0$$

at $g_P(x)$, which is automatically satisfied. □

As can be seen from the works of Mikhalkin [35] and Nishinou-Siebert [38], counting tropical curves with suitable multiplicities should be the same as counting holomorphic curves. For obvious reasons, we set the multiplicity of the tropical curve appeared in the above proposition to be one. Now the corresponding holomorphic curve is nothing but φ_a.

3.2 SYZ transformations for toric Fano manifolds

In this section, we will define the SYZ mirror transformation for the toric Fano manifold $\bar{X}$ as *a combination of the semi-flat SYZ transformation and taking fiberwise Fourier series.*

We equip the cover $\tilde{X}$ with the symplectic structure $\pi^*(\omega_X)$, which by abusing notations again, will still be denoted as ω_X. We also use μ to denote the fibration

$$\mu : \tilde{X} \to P.$$

Analog to the semi-flat case, consider the fiber product

$$\tilde{X} \times_P Y = P \times i(N \times T_N \times T_M)$$

of the fibrations $\mu : \tilde{X} \to P$ and $\nu : Y \to P$. Note that we have a covering map $\tilde{X} \times_P Y \to X \times_P Y$. Pulling back $F \in \Omega^2(X \times_P Y)$, we get the fiberwise universal curvature two-form on $\tilde{X} \times_P Y$ which we still denote by F. In terms of coordinates, we can write $F = i \sum_{j=1}^n dy_j \wedge du_j$, just as before. We further define the *holonomy function* $\mathrm{hol} : \tilde{X} \times_P Y \to \mathbb{C}$ by

$$\mathrm{hol}(x, v, u, y) = \mathrm{hol}_{\nabla_y}(v) = e^{-i\langle y,v\rangle}$$

for $(x, v, u, y) \in \tilde{X} \times_P Y = P \times i(N \times T_N \times T_M)$.

Definition 3.2.1. *The SYZ mirror transformation for toric Fano manifolds $\mathcal{F} : \Omega^*(\tilde{X}) \to \Omega^*(Y)$ is defined by*

$$\mathcal{F}(\alpha) = \pi_{Y,*}(\pi_{\tilde{X}}^*(\alpha) \wedge e^F hol) = \int_{N \times T_N} \pi_{\tilde{X}}^*(\alpha) \wedge e^F hol$$

where $\pi_{\tilde{X}} : \tilde{X} \times_P Y \to \tilde{X}$ and $\pi_Y : \tilde{X} \times_P Y \to Y$ are the two natural projections.

Before stating the basic properties of $\mathcal{F}$, we introduce a class of functions on $\tilde{X}$:

Definition 3.2.2. *A T_N-invariant function $f : \tilde{X} \to \mathbb{C}$ is said to be admissible if for any $(x, v, u) \in \tilde{X} = P \times i(N \times T_N)$,*

$$f(x, v, u) = f_v e^{-\langle x,v\rangle}$$

for some $f_v \in \mathbb{C}$, and the fibrewise Fourier series

$$\hat{f} := \sum_{v \in N} f_v e^{-\langle x,v\rangle} hol_{\nabla_y}(v) = \sum_{v \in N} f_v e^{-\langle z,v\rangle},$$

where $z = (z_1, \ldots, z_n) = x + iy$, is convergent and analytic.

For functions $f, g \in C^\infty(\tilde{X})$, we again define their *convolution product* $f \star g \in C^\infty(\tilde{X})$ by

$$f \star g(x, v, u) = \sum_{v_1 + v_2 = v} f(x, v_1, u) g(x, v_2, u).$$

If f, g are admissible, so is $f \star g$. Let $\mathcal{A}(X)$ be the ring of admissible functions on $\tilde{X}$ equipped with the product defined by convolution. Notice that $\Phi_q, \Psi_1, \ldots, \Psi_d \in C^\infty(P \times N)$, now regarded as functions on $\tilde{X}$, are all admissible.

Let $\phi \in \mathcal{O}(Y)$ be a holomorphic function on Y. Recall that $Y = P \times iT_M$. Restricting ϕ to a fiber $\nu^{-1}(x) = T_M$ gives a function ϕ_x on the torus $T_M = (T_N)^*$. Define $\hat{\phi} : \tilde{X} \to \mathbb{C}$ by

$$\hat{\phi}(x, v, u) = \hat{\phi}_x(v),$$

where $\hat{\phi}_x(v)$ is the Fourier coefficient of the function ϕ_x at $v \in N$. This is called the *fiberwise Fourier coefficients* of ϕ and it is clearly admissible. The following lemma follows from the theory of Fourier series.

Lemma 3.2.1. *Taking fiberwise Fourier series*

$$\mathcal{A}(X) \to \mathcal{O}(Y), \quad f \mapsto \hat{f}$$

is an isomorphism of rings and its inverse is given by taking fiberwise Fourier coefficients. In particular, $\hat{\hat{f}} = f$.

The basic properties of the SYZ mirror transformation are summarized in the following theorem.

Theorem 3.2.1. *Let* $\mathcal{A}(X)e^{\omega_X} := \{fe^{\omega_X} : f \in \mathcal{A}(X)\} \subset \Omega^*(\tilde{X})$ *and* $\mathcal{O}(Y)\Omega_Y := \{\phi\Omega_Y : \phi \in \mathcal{O}(Y)\} \subset \Omega^*(Y)$.

(i) For any $f \in \mathcal{A}(X)$,

$$\mathcal{F}(fe^{\omega_X}) = \hat{f}\Omega_Y \in \mathcal{O}(Y)\Omega_Y.$$

(ii) If we define the inverse SYZ mirror transformation $\mathcal{F}^{-1} : \Omega^*(Y) \to \Omega^*(\tilde{X})$ *by*

$$\mathcal{F}^{-1}(\alpha) = i^{-n}\pi_{\tilde{X},*}(\pi_Y^*(\alpha) \wedge e^{-F} hol^{-1}) = i^{-n} \int_{T_M} \pi_Y^*(\alpha) \wedge e^{-F} hol^{-1}$$

where $hol^{-1} : \tilde{X} \times_P Y \to \mathbb{C}$ *is the function defined by*

$$hol^{-1}(x, v, u, y) = hol_{\nabla_y}(v)^{-1} = e^{i\langle y,v\rangle},$$

then

$$\mathcal{F}^{-1}(\phi\Omega_Y) = \hat{\phi}e^{\omega_X}$$

for any $\phi \in \mathcal{O}(Y)$.

(iii) The restriction map $\mathcal{F} : \mathcal{A}(X)e^{\omega_X} \to \mathcal{O}(Y)\Omega_Y$ *is a bijection with inverse* $\mathcal{F}^{-1} : \mathcal{O}(Y)\Omega_Y \to \mathcal{A}(X)e^{\omega_X}$, *i.e.*

$$\mathcal{F}^{-1} \circ \mathcal{F} = Id_{\mathcal{A}(X)e^{\omega_X}},\ \mathcal{F} \circ \mathcal{F}^{-1} = Id_{\mathcal{O}(Y)\Omega_Y}.$$

Proof. Suppose $f \in \mathcal{A}(X)$ is given by $f(x, v, u) = f_v e^{-\langle x,v\rangle}$. Then, by observing that both $\pi^*_{\tilde{X}}(f)$ and hol are T_N-invariant, we have

$$\begin{aligned}
\mathcal{F}(fe^{\omega_X}) &= \int_{N\times T_N} \pi^*_{\tilde{X}}(fe^{\omega_X}) \wedge e^F \text{hol} \\
&= \sum_{v\in N} \pi^*_{\tilde{X}}(f) \cdot \text{hol} \int_{T_N} \pi^*_{\tilde{X}}(e^{\omega_X}) \wedge e^F \\
&= \left(\sum_{v\in N} \pi^*_{\tilde{X}}(f) \cdot \text{hol}\right)\left(\int_{T_N} \pi^*_X(e^{\omega_X}) \wedge e^F\right).
\end{aligned}$$

The last equality is due to the fact that $\pi^*_{\tilde{X}}(e^{\omega_X}) = \pi^*_X(e^{\omega_X})$ and e^F are independent of $v \in N$. By Proposition 2.2.1, we already have

$$\int_{T_N} \pi^*_X(e^{\omega_X}) \wedge e^F = \Omega_Y,$$

while the first factor is given by

$$\begin{aligned}
\sum_{v\in N} \pi^*_{\tilde{X}}(f) \cdot \text{hol} &= \sum_{v\in N} f_v e^{-\langle x,v\rangle} e^{-i\langle y,v\rangle} \\
&= \sum_{v\in N} f_v e^{-\langle z,v\rangle} \\
&= \hat{f}.
\end{aligned}$$

This proves (i). For (ii), expand $\phi \in \mathcal{O}(Y)$ into a relative Fourier series $\phi(z) = \sum_{w\in N} \hat{\phi}_x(w) e^{-i\langle y,w\rangle}$. Then

$$\begin{aligned}\mathcal{F}^{-1}(\phi\Omega_Y) &= i^{-n}\int_{T_M} \pi_Y^*(\phi\Omega_Y)\wedge e^{-F}\mathrm{hol}^{-1}\\ &= i^{-n}\sum_{w\in N}\left(\hat{\phi}_x(w)\int_{T_M} e^{i\langle y,v-w\rangle}\pi_Y^*(\Omega_Y)\wedge e^{-F}\right).\end{aligned}$$

Here comes the key observation: if $v - w \neq 0$, then, using (the proof of) the second part of Proposition 2.2.2, we have

$$\begin{aligned}\int_{T_M} e^{i\langle y,v-w\rangle}\pi_Y^*(\Omega_Y)\wedge e^{-F} &= \int_{T_M} e^{i\langle y,v-w\rangle}\left(\bigwedge_{j=1}^{n}(dx_j+idy_j)\right)\wedge e^{i\sum_{j=1}^n du_j\wedge dy_j}\\ &= i^n e^{\omega_X}\int_{T_M} e^{i\langle y,v-w\rangle} dy_1\wedge\ldots\wedge dy_n\\ &= 0.\end{aligned}$$

Hence,

$$\mathcal{F}^{-1}(\phi\Omega_Y) = i^{-n}\hat{\phi}_x(v)\int_{T_M}\pi_Y^*(\Omega_Y)\wedge e^{-F} = \hat{\phi}e^{\omega_X}.$$

(iii) follows from (i), (ii) and Lemma 3.2.1. □

From now on, we will also use $\mathcal{F}$ to denote the fiberwise Fourier series, i.e.

$$\mathcal{F} : \mathcal{A}(X) \to \mathcal{O}(Y),\ f \mapsto \hat{f},$$

which we regard as an SYZ mirror transformation. Which one we are referring to should be clear from the context. Theorem 1.2.1, which we recall as follows, is now a corollary of Theorem 3.2.1.

Theorem 3.2.2 (=Theorem 1.2.1)**.** *The SYZ transformation of the generating function Φ_q for holomorphic discs counting on the* *toric Fano manifold $\bar{X}$ gives the superpotential W on the mirror manifold $Y \cong (\mathbb{C}^*)^n$:*

$$\mathcal{F}(\Phi_q) = e^W \in \mathcal{O}(Y).$$

Furthermore, we can incorporate the symplectic structure ω_X on X to give the holomorphic volume form on the Landau-Ginzburg model (Y, W) in the sense that

$$\mathcal{F}(\Phi_q e^{\omega_X}) = e^W \Omega_Y.$$

Proof. Recall that

$$\Phi_q(x, v, u) = \sum_{\beta \in \pi_2^+(\bar{X}, L_x),\ \partial\beta = v} \frac{1}{w(\beta)} e^{-\int_\beta \omega_{\bar{X}}}.$$

For $\beta \in \pi_2^+(\bar{X}, L_x)$ with $\partial\beta = v$, we have $\int_\beta \omega_{\bar{X}} = \langle x, v\rangle + \text{const}$. So Φ_q is admissible. It remains to show that the fiberwise Fourier series $\mathcal{F}(\Phi_q) = \hat{\Phi}_q = e^W$. Remember that the superpotential W is given, in our coordinates, by $W = \sum_{i=1}^d e^{-\langle z, v_i\rangle + \lambda_i}$. For $z = x + iy$,

$$\begin{aligned}
\hat{\Phi}_q(z) &= \sum_{v \in N} \left(\sum_{\beta \in \pi_2^+(\bar{X}, L_x),\ \partial\beta = v} \frac{1}{w(\beta)} e^{-\int_\beta \omega_{\bar{X}}} \right) \mathrm{hol}_{\nabla_y}(v) \\
&= \sum_{k_1, \ldots, k_d \in \mathbb{Z}_{\geq 0}} \frac{1}{\prod_{i=1}^d k_i!} e^{-\sum_{i=1}^d k_i \int_{\beta_i} \omega_{\bar{X}}} e^{-\sum_{i=1}^d k_i i \langle y, v_i\rangle} \\
&= \prod_{i=1}^d \left(\sum_{k_i \in \mathbb{Z}_{\geq 0}} \frac{1}{k_i!} \left(e^{-\int_{\beta_i} \omega_{\bar{X}}} e^{-i\langle y, v_i\rangle}\right)^{k_i} \right) \\
&= \prod_{i=1}^d \exp\left(e^{-\langle z, v_i\rangle + \lambda_i}\right) \\
&= e^W.
\end{aligned}$$

$\square$

The form $\Phi_q e^{\omega_X}$ can be viewed as the symplectic structure weighted (or corrected) by holomorphic discs. That we call $e^W \Omega_Y$ the holomorphic volume form of the Landau-Ginzburg model (Y, W) can be justified in several ways. For instance, in the theory of singularities, one considers the complex oscillating integrals

$$I = \int_\Gamma e^W \Omega_Y$$

where Γ is some real n-dimensional cycle in Y constructed by the Morse theory of the function $\mathrm{Re}(W)$. These integrals resemble the periods of holomorphic volume forms on Calabi-Yau manifolds, and they satisfy certain Picard-Fuchs equations (see, for example, Givental [19]).

3.3 Quantum cohomology vs. Jacobian ring

The purpose of this section is to give a proof of Theorem 1.2.2. But before that, let's recall the definition of the Jacobian ring $Jac(W)$. Recall that the mirror manifold Y is given by

$$Y = \Big\{(Z_1, \ldots, Z_d) \in (\mathbb{C}^*)^d : \sum_{i=1}^d Q_{ia} Z_i = r_a,\ a = 1, \ldots, l\Big\}.$$

For convenience, from now on we will, instead of $Z_1, \ldots, Z_d$, use the exponential coordinates, i.e. we replace e^{-Z_i} by Z_i. Then Y can be written as

$$Y = \Big\{(Z_1, \ldots, Z_d) \in (\mathbb{C}^*)^d : \prod_{i=1}^d Z_i^{Q_{ia}} = q_a,\ a = 1, \ldots, l\Big\}$$

where $q_a = e^{-r_a}$, $a = 1, \ldots, l$, are the Kähler parameters as before. In these coordinates, the ring $\mathbb{C}[Y]$ of regular functions of Y is given by

$$\mathbb{C}[Y] = \mathbb{C}[Z_1, \ldots, Z_d] \Big/ \Big\langle \prod_{i=1}^d Z_i^{Q_{ia}} - q_a : a = 1, \ldots, l\Big\rangle,$$

where $\mathbb{C}[Z_1, \ldots, Z_d]$ is the ring of polynomials in d variables, and the superpotential $W : Y \to \mathbb{C}$ is just the sum of coordinates

$$W = Z_1 + \ldots + Z_d.$$

Similarly, we change the coordinates on Y by replacing e^{-z_j} by z_j, so that the equation $Z_i = \langle z, v_i \rangle - \lambda_i$ becomes

$$Z_i = e^{\lambda_i} z^{v_i} = e^{\lambda_i} z_1^{v_i^1} \ldots z_n^{v_i^n}$$

for $i = 1, \ldots, d$. In these coordinates, W is written as

$$W = e^{\lambda_1} z^{v_1} + \ldots + e^{\lambda_d} z^{v_d}$$

and $\mathbb{C}[Y] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$. Now, the Jacobian ring is defined as the quotient of $\mathbb{C}[Y]$ by the ideal generated by the logarithmic derivatives of W:

$$\begin{aligned} Jac(W) &= \mathbb{C}[Y] \Big/ \Big\langle z_j \frac{\partial W}{\partial z_j} : j = 1, \ldots, n \Big\rangle \\ &= \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \Big/ \Big\langle z_j \frac{\partial W}{\partial z_j} : j = 1, \ldots, n \Big\rangle. \end{aligned}$$

Theorem 1.2.2 follows from:

Theorem 3.3.1. *The induced transformation*

$$\mathcal{F} : \mathbb{C}[\Psi_1^{\pm 1}, \ldots, \Psi_n^{\pm 1}]/\mathcal{L} \to Jac(W)$$

is an isomorphism of $\mathbb{C}$-algebras. In particular, the SYZ mirror transformation induces a natural isomorphism of $\mathbb{C}$-algebras

$$\mathcal{F} : QH^*(\bar{X}) \xrightarrow{\cong} Jac(W)$$

provided that $\bar{X}$ is a product of projective spaces.

Proof. First of all, obvious that $\mathcal{F}(\Psi_i)$ is the monomial $Z_i = e^{\lambda_i} z^{v_i}$ for $i = 1, \ldots, d$. Indeed, in the new coordinates, if $f \in \mathcal{A}(X)$ is given by $f(x, v, u) = f_v e^{-\langle x, v \rangle}$, then

$$\mathcal{F}(f) = \hat{f} = \sum_{v \in N} f_v z^v.$$

Hence $\mathcal{F}(\Psi_i) = e^{\lambda_i} z^{v_i}$ and in particular $\mathcal{F}(\Psi_i) = z_i$ for $i = 1, \ldots, n$. Next, note that

$$z_j \frac{\partial W}{\partial z_j} = \sum_{i=1}^{d} z_j \frac{\partial}{\partial z_j} (e^{\lambda_i} z_1^{v_i^1} \ldots z_n^{v_i^n}) = \sum_{i=1}^{d} v_i^j e^{\lambda_i} z_1^{v_i^1} \ldots z_n^{v_i^n} = \sum_{i=1}^{d} v_i^j Z_i$$

for $j = 1, \ldots, n$. The inverse SYZ transformation of $z_j \frac{\partial W}{\partial z_j}$ is thus given by

$$\mathcal{F}^{-1}(z_j \frac{\partial W}{\partial z_j}) = \widehat{\sum_{i=1}^{d} v_i^j Z_i} = \sum_{i=1}^{d} v_i^j \Psi_i.$$

The theorem now follows from the fact that all linear equivalences are generated by the relations $\sum_{i=1}^{d} v_i^j \Psi_i = 0$ for $j = 1, \ldots, n$. $\square$

We give two examples to illustrate the SYZ mirror transformation and origination of quantum corrections.

Example 1: $\bar{X} = \mathbb{P}^2$. In this case, $N = \mathbb{Z}^2$. Let $\{e_1, e_2\}$ be the standard basis of N. The primitive generators of the fan of $\bar{X}$ are given by $v_1 = e_1 = (1, 0)$, $v_2 = e_2 = (0, 1)$ and $v_3 = -e_1 - e_2 = (-1, -1)$, and the polytope $\bar{P} \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ is defined by the inequalities

$$x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq t,$$

where $t > 0$. The mirror manifold is given by

$$Y = \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 : Z_1 Z_2 Z_3 = q\} \cong (\mathbb{C}^*)^2,$$

where $q = e^{-t}$ is the Kähler parameter, and the superpotential $W : Y \to \mathbb{C}$ can be written as

$$W = Z_1 + Z_2 + Z_3 = z_1 + z_2 + \frac{q}{z_1 z_2}$$

in the coordinates (z_1, z_2) of Y. By our derivation, the Jacobian ring $Jac(W)$ is then given by

$$\begin{aligned} Jac(W) &= \mathbb{C}[Z_1, Z_2, Z_3]/\langle Z_1 - Z_3, Z_2 - Z_3, Z_1 Z_2 Z_3 - q\rangle \\ &= \mathbb{C}[Z]/\langle Z^3 - q\rangle. \end{aligned}$$

There are three toric divisors D_1, D_2, D_3 corresponding to three admissible functions $\Psi_1, \Psi_2, \Psi_3 : \tilde{X} \to \mathbb{C}$ defined by

$$\begin{aligned} \Psi_1(x, v, u) &= \begin{cases} e^{-x_1} & \text{if } v = (1,0) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_2(x, v, u) &= \begin{cases} e^{-x_2} & \text{if } v = (0,1) \\ 0 & \text{otherwise,} \end{cases} \\ \Psi_3(x, v, u) &= \begin{cases} e^{-(t-x_1-x_2)} & \text{if } v = (-1,-1) \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

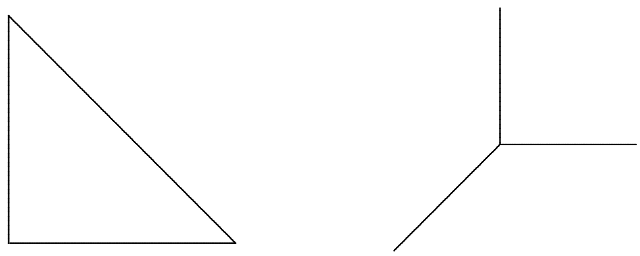
respectively. The quantum cohomology ring is given by

$$\begin{aligned} QH^*(\bar{X}) &= \mathbb{C}[D_1, D_2, D_3]/\langle D_1 - D_3, D_2 - D_3, D_1 * D_2 * D_3 - q\rangle \\ &= \mathbb{C}[H]/\langle H^3 - q\rangle \end{aligned}$$

where H is the hyperplane class. Corrections appear only in one relation, namely,

$$D_1 * D_2 * D_3 = q.$$

Fix a point $x \in P$. Then the correction is due to the unique holomorphic curve $\varphi : (\mathbb{P}^1; x_0, x_1, x_2, x_3) \to \bar{X}$ of degree 1 (i.e. a line) with 4 marked points such that $\varphi(x_0) = p$ and $\varphi(x_i) \in D_i$ for $i = 1, 2, 3$. The tropical curve corresponding to this line is Γ, which is glued from three half lines emanating from the point $g_P(x) \in N_{\mathbb{R}}$ in the directions $v_1 = (1, 0)$, $v_2 = (0, 1)$ and $v_3 = (-1, -1)$. These half lines are the tropical discs corresponding to the Maslov index two holomorphic discs φ_1, φ_2 and φ_3 which intersect at one point with the corresponding toric divisors D_1, D_2 and D_3 respectively and whose boundaries are mapped to the Lagrangian torus L_x. See Figure 1 below.



P Γ

Figure 1

Example 2: $\bar{X} = \mathbb{P}^1 \times \mathbb{P}^1$. Again let $\{e_1, e_2\}$ be the standard basis of $N = \mathbb{Z}^2$. The primitive generators of the fan of $\bar{X}$ are given by $v_{1,1} = e_1 = (1,0)$, $v_{2,1} = -e_1 = (-1,0)$, $v_{1,2} = e_2 = (0,1)$ and $v_{2,2} = -e_2 = (0,-1)$, and the polytope $\bar{P} \subset M_{\mathbb{R}} \cong \mathbb{R}^2$ is defined by the inequalities

$$0 \leq x_1 \leq t_1,\ 0 \leq x_2 \leq t_2$$

where $t_1, t_2 > 0$. The mirror Landau-Ginzburg model (Y, W) consists of

$$Y = \{(Z_{1,1}, Z_{2,1}, Z_{1,2}, Z_{2,2}) \in \mathbb{C}^4 : Z_{1,1}Z_{2,1} = q_1, Z_{2,1}Z_{2,2} = q_2\} \cong (\mathbb{C}^*)^2,$$

where $q_1 = e^{-t_1}$ and $q_2 = e^{-t_2}$ are the Kähler parameters, and

$$W = Z_{1,1} + Z_{2,1} + Z_{1,2} + Z_{2,2} = z_1 + \frac{q_1}{z_1} + z_2 + \frac{q_2}{z_2}.$$

The Jacobian ring $Jac(W)$ is then given by

$$\begin{aligned} Jac(W) &= \frac{\mathbb{C}[Z_{1,1}, Z_{2,1}, Z_{1,2}, Z_{2,2}]}{\langle Z_{1,1} - Z_{2,1}, Z_{1,2} - Z_{2,2}, Z_{1,1}Z_{2,1} - q_1, Z_{1,2}Z_{2,2} - q_2\rangle} \\ &= \mathbb{C}[X, Y]/\langle X^2 - q_1, Y^2 - q_2\rangle. \end{aligned}$$

There are four toric divisors $D_{1,1}, D_{2,1}, D_{1,2}, D_{2,2}$ corresponding to the admissible functions $\Psi_{1,1}, \Psi_{2,1}, \Psi_{1,2}, \Psi_{2,2} : \tilde{X} \to \mathbb{C}$ defined by

$$
\begin{aligned}
\Psi_{1,1}(x,v,u) &= \begin{cases} e^{-x_1} & \text{if } v=(1,0) \\ 0 & \text{otherwise,} \end{cases} \\
\Psi_{2,1}(x,v,u) &= \begin{cases} e^{-(t_1-x_1)} & \text{if } v=(0,-1) \\ 0 & \text{otherwise,} \end{cases} \\
\Psi_{1,2}(x,v,u) &= \begin{cases} e^{-x_2} & \text{if } v=(0,1) \\ 0 & \text{otherwise,} \end{cases} \\
\Psi_{2,2}(x,v,u) &= \begin{cases} e^{-(t_2-x_2)} & \text{if } v=(0,-1) \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

respectively. The quantum cohomology is given by

$$
\begin{aligned}
QH^*(\bar{X}) &= \frac{\mathbb{C}[D_{1,1},D_{2,1},D_{1,2},D_{2,2}]}{\langle D_{1,1}-D_{2,1}, D_{1,2}-D_{2,2}, D_{1,1}*D_{2,1}-q_1, D_{1,2}*D_{2,2}-q_2\rangle} \\
&= \mathbb{C}[H_1,H_2]/\langle H_1^2-q_1, H_2^2-q_2\rangle
\end{aligned}
$$

where H_1 and H_2 are pullbacks of the hyperplane classes in the first and second factors respectively. The corrections appear in two relations

$$D_{1,1}*D_{2,1}=q_1 \text{ and } D_{1,2}*D_{2,2}=q_2.$$

We will consider only the first one. For any $x \in P$, there are two Maslov index two holomorphic discs $\varphi_{1,1}$ and $\varphi_{2,1}$, each intersects at one interior point with the corresponding divisor, and whose boundaries are mapped to L_x. In this case, the two holomorphic discs glue together *directly* to give the holomorphic curve $\varphi_1 : (\mathbb{P}^1 : x_0, x_1, x_2) \to \bar{X}$, which is of degree 1 (again a line) and passes through L_x, $D_{1,1}$, $D_{2,1}$. φ_1 accounts for the quantum correction involved in the relation $D_{1,1}*D_{2,1}=q_1$.

3.4 Correspondences of cycles

This section is an attempt to understand the correspondences between A-branes on a toric Fano manifold $\bar{X}$ and B-branes on the mirror Landau-Ginzburg model (Y, W) using the SYZ framework.

We will deal with the simplest case of the correspondence. Let $L_x = \mu_{\bar{X}}^{-1}(x)$ be a Lagrangian torus fiber of $\bar{X}$ over $x \in P$ equipped with a flat $U(1)$-bundle $\mathbb{L}_y = (L \times \mathbb{C}, \nabla_y)$ where as before, ∇_y is a flat $U(1)$-connection on the trivial line bundle over L_x. This gives us an A-brane $(L_x, \mathbb{L}_y)$. According to the SYZ Mirror Conjecture, the mirror of this A-brane is given by the B-brane $(z = x + iy \in Y, \mathcal{O}_z)$. In other words, the correspondence between the objects is the same as in the semi-flat case. The difference emerges when we consider morphisms.

According to Hori (see [27], Chapter 39.), the endomorphism algebra $\mathrm{End}(z, \mathcal{O}_z)$ of the B-brane $(z, \mathcal{O}_z)$ is given by the cohomology of the complex

$$(Cl(T_zY, Hess(W)_z), \delta = \iota_{\partial W_z})$$

where $Cl(T_zY, Hess(W)_z)$ denotes the Clifford algebra generated by the tangent space T_zY equipped with the bilinear form given by the Hessian of W at the point z, and $\iota_{\partial W_z}$ is the contraction with the vector ∂W_z. The following elementary proposition shows that the introduction of the superpotential W "localizes" the category B-type D0-branes to the critical locus of W.

Proposition 3.4.1. *The endomorphism algebra End$(z, \mathcal{O}_z)$ is nontrivial if and only if $z \in Y$ is a critical point of the superpotential $W : Y \to \mathbb{C}$. In this case, End$(z, \mathcal{O}_z) = Cl(T_zY, Hess(W)_z)$.*

On the other hand, the endomorphism algebra of the A-brane $(L_x, \mathbb{L}_y)$ is given by the *Floer cohomology ring*[1] $HF(L_x, \mathbb{L}_y)$,

[1] To avoid technicalities, we use $\mathbb{C}$ as the coefficient ring, instead of the Novikov ring Λ.

which is in turn, as a module, given by the cohomology of the complex [13], [12]

$$(H^*(L_x,\mathbb{C}) = \bigwedge^* H^1(L_x,\mathbb{C}), \delta = m_1)$$

where $m_1 = m_1(L_x,\mathbb{L}_y)$ denotes the Floer differential. In [13], [12], Cho and Oh explicitly computed the Floer differential m_1. Recall that $H^1(L_x,\mathbb{C})$ is canonically isomorphic to T_zY. Let $C_1,\ldots,C_n$ be the basis of $H^1(L_x,\mathbb{C})$ corresponding to $(z_1\frac{\partial}{\partial z_1})_z$, ..., $(z_n\frac{\partial}{\partial z_n})_z$. Then the intersection number $C_j\cdot\partial\beta_i = v_i^j$ and

$$m_1(C_{j_1}\wedge\ldots\wedge C_{j_k}) = \sum_{i=1}^d m_{1,\beta_i}(C_{j_1}\wedge\ldots\wedge C_{j_k})e^{-\int_{\beta_i}\omega_X}\mathrm{hol}_{\nabla_y}(\partial\beta_i)$$

where $m_{1,\beta_i}(C_{j_1}\wedge\ldots\wedge C_{j_k}) = \sum_{\alpha=1}^k(-1)^{k-1}(C_{j_\alpha}\cdot\partial\beta_i)C_{j_1}\wedge\ldots\wedge\widehat{C_{j_\alpha}}\wedge\ldots\wedge C_{j_k}$. It follows that m_1 coincides with $\iota_{\partial W_z}$ under the canonical isomorphism

$$H^*(L_x,\mathbb{C})\cong\bigwedge^* T_zY$$

which maps $C_{j_1}\wedge\ldots\wedge C_{j_k}$ to $(z_{j_1}\frac{\partial}{\partial z_{j_1}})_z\wedge\ldots\wedge(z_{j_k}\frac{\partial}{\partial z_{j_k}})_z$. Hence $HF(L_x,\mathbb{L}_y)$ is isomorphic to $\mathrm{End}(z,\mathcal{O}_z)$ as vector spaces. Moreover, Cho [12] proved that the Floer cohomology ring $HF(L_x,\mathbb{L}_y)$, equipped with the product structure given by $m_2 = m_2(L_x,\mathbb{L}_y)$, has a Clifford algebra structure generated by $H^1(L_x,\mathbb{C})$ with the bilinear form given by $Q(C_j,C_k) = \sum_{i=1}^d v_i^j v_i^k e^{-\int_{\beta_i}\omega_X}\mathrm{hol}_{\nabla_y}(\partial\beta_i)$. We conclude that

Proposition 3.4.2 (=Proposition 1.2.3)**.** *The Floer cohomology $HF(L_x,\mathbb{L}_y)$ is isomorphic to End$(z,\mathcal{O}_z)$ as $\mathbb{C}$-algebras. In particular, $HF(L_x,\mathbb{L}_y)$ is nontrivial if and only if $z = x+iy\in Y$ is a critical point of $W : Y\to\mathbb{C}$.*

□ **End of chapter.**

Chapter 4

Future directions

The work in this thesis represents the first step in our program which is aimed at exploring mirror symmetry via SYZ mirror transformations. Our results showed that these transformations can indeed be applied successfully to explain how quantum corrections arise. There are certainly much more work remains to be done in the future. In this final chapter, we will comment on several possible future research directions.

Toric Fano manifolds

We have seen that the simplest correspondence between A-branes on a toric Fano manifold $\bar{X}$ and B-branes on the mirror Landau-Ginzburg model (Y, W), namely

$$(L_x, \mathbb{L}_y) \longleftrightarrow (z, \mathcal{O}_z),$$

is compatible with the SYZ philosophy. It is desirable to see how other A-branes on X are transformed to B-branes on (Y, W). An interesting and important example would be the Lagrangian submanifold $\mathbb{RP}^n \subset \mathbb{CP}^n$, which can be viewed as a multi-section of the moment map of $\mathbb{CP}^n$, equipped with the trivial flat $U(1)$-bundle. Employing the SYZ approach, The mirror B-brane is expected to be a rank-2^n holomorphic vector bundle over Y, equipped with some additional information related to W. A possible choice of this additional information would be a *matrix factorization* of W; currently, it is widely believed that the category of B-branes on (Y, W) is given by the category of matrix factoriztions of W (this was first proposed by Kontsevich, see Orlov [39] for details). The relation between these matrix factorizations and the computation of Floer cohomology will be the key to a complete understanding of the correspondences.

On the other hand, we have not even touch the correspondence between B-branes on $\bar{X}$ and A-branes on (Y, W). As we mentioned in the introduction, the results of Seidel [40], Ueda [44], Auroux-Katzarkov-Orlov [6], [7] and Abouzaid [1], [2] have provided substantial evidences for this half of the Homological Mirror Symmetry Conjecture. In particular, Abouzaid [2] made use of an idea originated from the SYZ Mirror Conjecture, namely, the mirror of a Lagrangian section should be a holomorphic line bundle. His results also showed that the correspondence should be in line with the SYZ picture. It is an interesting question whether one can identify B-branes on $\bar{X}$ directly with A-branes on (Y, W) using SYZ fibrations and SYZ mirror transformations. In particular, we would like to identify geometrically the SYZ mirrors of Hermitian-Yang-Mills connections on holomorphic bundles over $\bar{X}$.

Toric non-Fano or non-toric Fano manifolds

In our work, we made heavy use of Lagrangian torus fibration provided by the moment map associated to the Hamiltonian T^n-action on a toric Fano manifold $\bar{X}$. This is not available in the case of non-toric Fano manifolds (e.g. Grassmanians, flag varieties). While the mirror symmetry for these non-toric Fano manifolds has been studied for some time by Givental [17] and others, new tools and new ideas are needed if we want to apply SYZ mirror transformations to understand the quantum corrections in these cases.

On the other hand, the mirror symmetry for toric non-Fano manifolds is also not understood well. As can be seen from the works of Givental [21], [18], the mirror map between the Kähler and complex moduli spaces in this case is a nontrivial *coordinate change*, instead of an identity map as in the toric Fano case. Hence, the definitions of the SYZ mirror transformations may have to be adjusted to incorporate the nontrivial mirror map. In this respect, also deserve investigation would be Fano complete intersections in toric varieties.

Calabi-Yau manifolds

The ultimate goal of our program is no doubt the mirror symmetry for Calabi-Yau manifolds and the SYZ Mirror Conjecture. Works of Fukaya [15], Kontsevich-Soibelman [31] and Gross-Siebert [24] have laid a very solid foundation for understanding the mirror symmetry for Calabi-Yau manifolds. In view of the fact that toric varieties have played an important role in the constructions of Gross and Siebert, it would nice if we can incorporate our methods with their new techniques to study SYZ mirror transformations for Calabi-Yau manifolds; and hopefully, this would let us reveal the secret of mirror symmetry for Calabi-Yau manifolds.

□ **End of chapter.**

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