On Some Constructions of Calabi-Yau Manifolds

CHAN Kwok Wai

A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
Master of Philosophy

in

Mathematics

June 2004

Abstract

Classical examples of $K3$ surfaces are given by hypersurfaces and complete intersections in projective spaces, double coverings of $\mathbb{P}^2$ branched along sextic curves and the famous Kummer surfaces. In this report, we will consider the higher dimensional analog of these constructions, namely, of Calabi-Yau manifolds. Firstly we construct Calabi-Yau manifolds as hypersurfaces and complete intersections in toric varieties, following Batyrev and Borisov. This generalizes Calabi-Yau hypersurfaces and complete intersections in projective spaces, product of projective spaces and weighted projective spaces. Next we construct Calabi-Yau manifolds by quotient. In particular, we look into the possibility of generalizing the Kummer construction of $K3$ surfaces, following Roan. Finally, we construct Calabi-Yau manifolds by coverings. The emphasis is on the construction of Calabi-Yau 3-folds as double coverings of $\mathbb{P}^3$ branched along octic surfaces, following Cynk. Moreover we will try to investigate further properties, e.g. modularity, of the Calabi-Yau manifolds we constructed.

摘要

K3 曲面的經典例子包括投影空間中的超曲面及完全交，沿六次曲線分歧的二維投影空間的二重覆蓋，及有名的 Kummer 曲面。於本報告中，我們將考慮這些構造的高維推廣，即 Calabi-Yau 流形的構造。首先我們參照 Batyrev 與 Borisov 的做法，把 Calabi-Yau 構作成環簇(toric varieties)中的超曲面及完全交，這推廣了投影空間,投影空間的積及權投影空間(weighted projective spaces)中的 Calabi-Yau 超曲面及完全交。然後，我們把 Calabi-Yau 流形構作成商，其中，我們會依照 Roan 的想法，探討推廣 Kummer 曲面的構造的可能性。最後，我們用覆蓋來構造 Calabi-Yau 流形，並跟隨 Cynk，重點考慮三維投影空間的沿八次曲面分歧的二重覆蓋。另外，我們亦會嘗試研究我們所構造的 Calabi-Yau 流形的其他性質，例如模性。

ACKNOWLEDGMENTS

I wish to express my gratitude to my supervisors Professor Shing-Tung Yau and Professor Kang Zuo for their helpful advices and valuable suggestions for the preparation of this thesis. I would like to particularly thank Professor Zuo for introducing me the fantastic world of algebraic geometry and for the numerous useful discussions during my study. I am excited by his mathematical knowledge and his brilliant ideas in attacking problems. Without his continual help and encouragement, I can hardly complete this thesis.

During my study, I have learnt much from Professor Hélène Esnault, Professor Brendan Hassett, Professor Sheng-Li Tan and Professor Eckart Viehweg. I am also grateful to Dr. Siu-Hung Tang for his continual hearty support and to Professor Bert van Geemen for his generous help and advices through emails.

Thanks also to The Department of Mathematics and The Institute of Mathematical Science of The Chinese University of Hong Kong for giving me a chance to study in a stimulating environment.

I would also like to thank my colleagues Mr. Yong-Dong Huang, Chang-Zheng Li, Jia-Jin Zhang, Yi Zhang and especially Mao Sheng for many useful discussions and helpful advices and for making these years so enjoyable.

I am indebted to my parents, my sister and my brother for their love and support throughout my life.

This thesis is dedicated to Dawn, for her continuous love, encouragement and understanding.

Contents

Introduction

An n-dimensional *Calabi-Yau manifold* is a compact Kähler manifold V of complex dimension n such that the canonical bundle K_X is trivial and $h^{k,0}(V) = dim(H^0(V, \Omega_V^k)) = 0$ for $1 < k < n$. In the past two decades, Calabi-Yau manifolds (or varieties) have continued to be a topic of intensive research. Motivations for such an investigation came from both the mathematical and physical sides. For mathematicians, one of the main reasons for studying Calabi-Yau varieties is that they are a key ingredient in *The Mori Program*, a vast project which is designed to accomplish the classification of all complex projective varieties. On the other hand, the interest of physicists mainly came from the connection between Calabi-Yau 3-folds and the so-called *Superstring Theory*. One of the many far-reaching consequences of this connection is a duality for Calabi-Yau 3-folds called *The Mirror Symmetry*. This, in particular, asserts that for any Calabi-Yau 3-fold V, there exists another Calabi-Yau 3-fold V', called the mirror of V; and that the Hodge numbers of V and V' satisfy the equalities:

$$h^{1,1}(V) = h^{2,1}(V'),\ h^{2,1}(V) = h^{1,1}(V')$$

These are very surprising and incredible from the mathematical viewpoint. Nevertheless, there has been increasing evidence in support of The Mirror Symmetry.

To test The Mirror Symmetry or The Mori Program or any other theories concerning Calabi-Yau manifolds, we should have enough examples of Calabi-Yau manifolds. Hence in this report, we are going to give some constructions of Calabi-Yau manifolds. Following the classical examples of $K3$ surfaces, which should be considered as 2-dimensional Calabi-Yau manifolds, we shall construct Calabi-Yau manifolds first as *hypersurfaces and complete intersections in toric varieties* in Chapter 2. This generalizes hypersurfaces and complete intersections in projective spaces, product of projective spaces and weighted projective spaces.

Then in Chapter 3, we shall consider the higher dimensional analog of Kummer surfaces, constructing Calabi-Yau manifolds by taking *quotients*. This will give examples of *rigid* Calabi-Yau 3-folds with *complex multiplications*. The final Chapter 4 is concerned with exhibiting Calabi-Yau manifolds as *coverings*. Our focus is on double coverings of $\mathbb{P}^3$ branched along octic surfaces and we will try to discuss the *modularity* of these double octics. The first chapter on toric geometry is mainly for reference and notation fixing.

Chapter 1

Introduction to Toric Geometry

In this chapter we present some results in toric geometry, which will be of use later. They are stated without proofs. For details please refer to [14], [19], [25].

1.1 Definitions of Toric Varieties

First we recall the definition of toric varieties. Let $N \cong \mathbb{Z}^n$ be a lattice (i.e. a free $\mathbb{Z}$-module) of rank $n \geq 1$; $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual and $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ the dual pairing.

Definition 1.1.1. *Let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Then the dual pairing naturally extends to $\langle \cdot, \cdot \rangle : N_{\mathbb{R}} \times M_{\mathbb{R}} \to \mathbb{R}$.*
(i) $\sigma \subset N_{\mathbb{R}}$ is called a convex rational polyhedral cone if there exist $v_1, \dots, v_s \in N$ such that

$$\sigma = \{\lambda_1 v_1 + \dots + \lambda_s v_s : \lambda_i \geq 0 \text{ for all } i\},$$

or if $\sigma = \{0\}$. σ is called strongly convex if $\sigma \cap (-\sigma) = \{0\}$. We also define the dimension of σ to be the dimension of the linear space $\mathbb{R} \cdot \sigma = \sigma + (-\sigma)$.
(ii) If $\sigma \subset N_{\mathbb{R}}$ is a convex rational polyhedral cone, then the dual cone $\check{\sigma} \subset M_{\mathbb{R}}$ is the set

$$\check{\sigma} := \{u \in M_{\mathbb{R}} : \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\},$$

which is also a convex rational polyhedral cone.

(iii) A face τ of a convex rational polyhedral cone σ is a subset

$$\tau = \sigma \cap u^{\perp} := \{v \in \sigma : \langle u, v \rangle = 0\}$$

for some $u \in \check{\sigma} \cap M$, and is written as $\tau \prec \sigma$. τ is a convex rational polyhedral cone if σ is so. A face of a face is a face, and any intersection of faces is a face. Also if $\tau \prec \sigma$ then $\check{\sigma} \cap \tau^{\perp}$ is a face of $\check{\sigma}$, with

$$dim(\tau) + dim(\check{\sigma} \cap \tau^{\perp}) = dim(N_{\mathbb{R}}) = n.$$

This gives a one-to-one correspondence between the faces of σ and the faces of $\check{\sigma}$.

Now *Gordan's Lemma* states that if σ is a convex rational polyhedral cone, the set S_σ defined by

$$S_\sigma := \check{\sigma} \cap M,$$

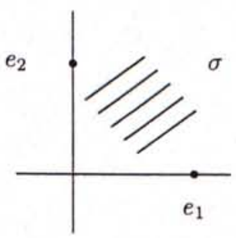
is a finitely generated semigroup. This gives a $\mathbb{C}$-algebra $A_\sigma := \mathbb{C}[S_\sigma]$. We can then define the *affine toric variety* associated to σ by:

$$U_\sigma := \text{Spec } \mathbb{C}[S_\sigma].$$

In particular for $\sigma = \{0\}$, $S_{\{0\}} = M$. We then have

$$\begin{aligned} U_{\{0\}} &= \text{Spec } \mathbb{C}[M] = \text{Spec } \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \\ &\cong \mathbb{C}^* \times \ldots \times \mathbb{C}^* = (\mathbb{C}^*)^n, \end{aligned}$$

which is an *affine algebraic torus*. As another example, take $N = \mathbb{Z}^2$ and $\{e_1, e_2\}$ the standard basis. Then $N_{\mathbb{R}} = \mathbb{R}^2$. Let σ be the cone generated by $\{e_1, e_2\}$.



Then

$$U_\sigma = \text{Spec } A_\sigma = \text{Spec } \mathbb{C}[X_1, X_2] \cong \mathbb{C}^2.$$

If $\tau \prec \sigma$ is a face, then $A_\sigma \subset A_\tau$. This determines a morphism $U_\tau = \text{Spec } A_\tau \to U_\sigma = \text{Spec } A_\sigma$, which can be shown to be a *principal open embedding*. In particular, if σ is strongly convex, $\{0\} \subset \sigma$ is a face, so that $T_N := (\mathbb{C}^*)^n$ is embedded as a principal open subset in U_σ.

To define general toric varieties, we need the notion of a *fan*:

Definition 1.1.2. *A fan Σ in $N_\mathbb{R}$ is a finite set of strongly convex rational polyhedral cones in $N_\mathbb{R}$ satisfying the following conditions:*
(1) If $\sigma \in \Sigma$, then $\tau \in \Sigma$ for each $\tau \prec \sigma$; and
(2) If σ, $\sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of both.
We define the support of Σ to be the set $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \subset N_\mathbb{R}$. Also for each $d \geq 0$, $\Sigma(d)$ denotes the set of d-dimensional cones of Σ.

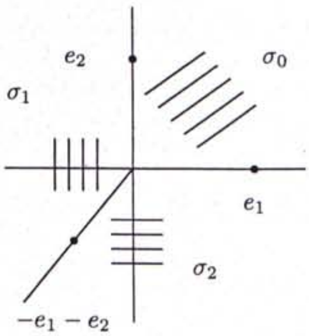
Definition 1.1.3. *Let N and M be as before and Σ be a fan in $N_\mathbb{R}$. The toric variety X_Σ defined by Σ is the identification space*

$$X_\Sigma = \coprod_{\sigma \in \Sigma} U_\sigma / \sim$$

by gluing together U_σ and $U_{\sigma'}$ along $U_{\sigma \cap \sigma'}$ which is embedded in both U_σ and $U_{\sigma'}$ as a principal open subset. X_Σ can be shown to be an irreducible, separated and normal (in fact Cohen-Macaulay) algebraic variety of dimension n.

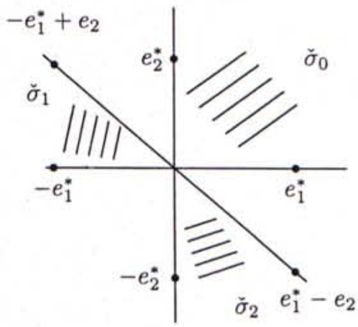
For example, again take $N = \mathbb{Z}^2$ and $\{e_1, e_2\}$ the standard basis. Consider the fan Σ in $N_\mathbb{R} = \mathbb{R}^2$ generated by the cones σ_0, σ_1 and σ_2 where

$$\begin{aligned} \sigma_0 &= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0} e_2, \\ \sigma_1 &= \mathbb{R}_{\geq 0} e_2 + \mathbb{R}_{\geq 0}(-e_1 - e_2), \\ \sigma_2 &= \mathbb{R}_{\geq 0} e_1 + \mathbb{R}_{\geq 0}(-e_1 - e_2). \end{aligned}$$



Using the dual basis $\{e_1^*, e_2^*\}$ of M, the dual cones are given by

$$\begin{aligned}\check{\sigma}_0 &= \mathbb{R}_{\geq 0}e_1^* + \mathbb{R}_{\geq 0}e_2^*,\\ \check{\sigma}_1 &= \mathbb{R}_{\geq 0}(-e_1^*) + \mathbb{R}_{\geq 0}(-e_1^* + e_2^*),\\ \check{\sigma}_2 &= \mathbb{R}_{\geq 0}(-e_2^*) + \mathbb{R}_{\geq 0}(e_1^* - e_2^*).\end{aligned}$$



So we have

$$\begin{aligned}U_{\sigma_0} &= \text{Spec } \mathbb{C}[X_1, X_2] \cong \mathbb{C}^2,\\ U_{\sigma_1} &= \text{Spec } \mathbb{C}[X_1^{-1}, X_1^{-1}X_2] \cong \mathbb{C}^2,\\ U_{\sigma_2} &= \text{Spec } \mathbb{C}[X_2^{-1}, X_1X_2^{-1}] \cong \mathbb{C}^2.\end{aligned}$$

If $(T_0 : T_1 : T_2)$ denotes the homogeneous coordinates on $\mathbb{P}^2$ and we let $X_1 = T_1/T_0$, $X_2 = T_2/T_0$, then X_Σ, which is given by U_{σ_0}, U_{σ_1} and U_{σ_2} gluing together, is isomorphic to $\mathbb{P}^2$.

Remark 1.1.1. *(i) Any 1-dimensional toric variety is isomorphic one of the followings:* $\mathbb{C}^*$, $\mathbb{C}$ *and* $\mathbb{P}^1$.

(ii) If Σ consists of just one n-dimensional cone σ and its faces, then X_Σ is just the n-dimensional affine toric variety U_σ.
(iii) If Σ is a fan in $N_\mathbb{R}$ and Σ' is a fan in $N'_\mathbb{R}$, then the set of products $\sigma \times \sigma'$, where $\sigma \in \Sigma$ and $\sigma' \in \Sigma'$, forms a fan $\Sigma \times \Sigma'$ in $N_\mathbb{R} \oplus N'_\mathbb{R}$ and

$$X_{\Sigma \times \Sigma'} = X_\Sigma \times X_{\Sigma'}.$$

This can of course be generalized to a product of any number of spaces.

Note that $T_N \cong (\mathbb{C}^*)^n$ is embedded as an open and dense subset in X_Σ (so that all toric varieties are birational to each other and in particular they are all rational). The action of T_N on itself naturally extends to an action of T_N on X_Σ:

$$\begin{array}{ccc} T_N \times X_\Sigma & \longrightarrow & X_\Sigma \\ \| \quad \cup & & \cup \\ T_N \times T_N & \longrightarrow & T_N \end{array}$$

This is why we call them *toric* varieties.
Let Σ be a fan in N and Σ' a fan in N'. Suppose $\varphi : N' \to N$ is a homomorphism of lattices whose scalar extension $\varphi : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies the following condition:

$$\text{for each cone } \sigma' \in \Sigma', \text{ there exists a cone } \sigma \in \Sigma \text{ with } \varphi(\sigma') \subset \sigma. \tag{1.1}$$

Then φ induces an equivariant morphism

$$\varphi_* : X_{\Sigma'} \to X_\Sigma.$$

1.2 Properties of Toric Varieties

One of the features of toric varieties is that their geometric properties can easily be translated into combinatorial properties of fans and cones. This makes the study of toric varieties more interesting and accessible. We will see some of these properties in the next few sections.

1.2.1 Smoothness

We begin with the characterization of smoothness of toric varieties:

Proposition 1.2.1. *An affine toric variety U_σ is nonsingular if and only if σ is generated by part of a $\mathbb{Z}$-basis of N. In this case,*

$$U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, \qquad \text{where } k = dim(\sigma).$$

More generally, a toric variety X_Σ is nonsingular if and only if Σ is nonsingular, in the sense that each $\sigma \in \Sigma$ is generated by part of a $\mathbb{Z}$-basis of N.

We can also determine when a toric variety is an *orbifold*, i.e. with only quotient singularities:

Proposition 1.2.2. *A toric variety X_Σ is an orbifold if and only if Σ is simplicial, i.e. each cone $\sigma \in \Sigma$ is generated by $\mathbb{R}$-linearly independent elements in $N_\mathbb{R}$.*

1.2.2 Compactness

Instead of discussing the compactness of a single toric variety, we consider the relative question, i.e. properness of maps between toric varieties. Recall that if Σ, Σ' are fans in N and N' respectively. Then a lattice homomorphism $\varphi : N' \to N$ whose scalar extension $\varphi : N'_\mathbb{R} \to N_\mathbb{R}$ satisfies (1.1) induces an equivariant morphism $\varphi_* : X_{\Sigma'} \to X_\Sigma$.

Proposition 1.2.3. *$\varphi_* : X_{\Sigma'} \to X_\Sigma$ is proper if and only if*

$$\varphi^{-1}(|\Sigma|) = |\Sigma'|.$$

In particular, applying this proposition to the zero map we have the

Corollary 1.2.1. *Let Σ be a fan in $N_\mathbb{R}$. Then the toric variety X_Σ is compact if and only if $|\Sigma| = N_\mathbb{R}$.*

On the other hand, concerning resolution of singularities, we have the following

Corollary 1.2.2. *$\varphi_* : X_{\Sigma'} \to X_\Sigma$ is proper and birational if and only if $\varphi : N' \to N$ is an isomorphism and Σ' is a locally finite subdivision of Σ under the identification $N'_{\mathbb{R}} = N_{\mathbb{R}}$, i.e. for each $\sigma \in \Sigma$, $\{\sigma' \in \Sigma' : \sigma' \subset \sigma\}$ is finite and σ is the union of such σ''s.*

From now on, unless otherwise stated, we assume all toric varieties are compact, i.e. $|\Sigma| = N_{\mathbb{R}}$

1.2.3 Stratification

Since each toric variety X_Σ admits a torus action of $T_N \cong (\mathbb{C}^*)^n$, we can decompose X_Σ into a disjoint union of T_N-invariant orbits. For each $\tau \in \Sigma$ let N_τ be the sublattice of N generated by $\tau \cap N$, and

$$N(\tau) = N/N_\tau,\ M(\tau) = \tau^\perp \cap M$$

the quotient and its dual. $N(\tau)$ can be proved to be a lattice, so we can define an affine toric variety

$$O_\tau := T_{N(\tau)} = \mathrm{Hom}(M(\tau), \mathbb{C}^*) = \mathrm{Spec}\ \mathbb{C}[M(\tau)] = N(\tau) \otimes_{\mathbb{Z}} \mathbb{C}^*$$

of dimension $n - dim(\tau)$. If we define

$$Star(\tau) := \{\bar{\sigma} \subset N(\tau)_{\mathbb{R}} : \sigma \succ \tau\}$$

where $\bar{\sigma}$ denotes the image of σ under the projection $N_{\mathbb{R}} \to N(\tau)_{\mathbb{R}}$, then $Star(\tau)$ is a fan in $N(\tau)_{\mathbb{R}}$ and the closure of O_τ is given by the $(n - dim(\tau))$-dimensional toric variety

$$V_\tau := X_{Star(\tau)}.$$

We have the

Proposition 1.2.4. *Among the orbits O_τ, orbit closures V_τ and the affine open sets U_σ, the following relations hold:*

(i) $U_\sigma = \coprod_{\tau \prec \sigma} O_\tau;$

(ii) $V_\tau = \coprod_{\gamma \succ \tau} O_\gamma;$

(iii) $O_\tau = V_\tau - \bigcup_{\gamma \supsetneq \tau} V_\gamma.$

In particular, X_Σ is a disjoint union of the O_τ's, i.e. $X_\Sigma = \coprod_{\tau \in \Sigma} O_\tau$.

1.3 Divisors on Toric Varieties

The discussion is restricted to T_N-invariant divisors.

1.3.1 Weil divisors

By the stratification of X_Σ described above, we know that the T_N-invariant prime divisors of X_Σ are given by the closures V_τ of the 1-dimensional orbits O_τ, $\tau \in \Sigma(1)$. Number the 1-dimensional cones as $\tau_1, ..., \tau_d$, and let v_i, $i = 1, ..., d$ be the primitive generator of τ_i. Then the prime divisors are given as:

$$D_i := V_{\tau_i},$$

and the T_N-invariant Weil divisors are formal sums $\sum_{i=1}^d a_i D_i$, $a_i \in \mathbb{Z}$.

1.3.2 Cartier divisors

To deal with (T_N-invariant)Cartier divisors, we first introduce the so-called support functions.

Definition 1.3.1. *A continuous piecewise linear function $h : |\Sigma| = N_\mathbb{R} \to \mathbb{R}$ is called an integral support function if for each cone $\sigma \in \Sigma$ there exists $u(\sigma) \in M$ such that*

$$h(v) = \langle u(\sigma), v \rangle \qquad \textit{for } v \in \sigma.$$

As is well known, we then have the following

Proposition 1.3.1. *T_N-invariant Cartier divisors D on X_Σ are in one-to-one correspondence with integral support functions $h : N_\mathbb{R} \to \mathbb{R}$.*

If $h : N_\mathbb{R} \to \mathbb{R}$ is an integral support function, we denote by D_h the corresponding Cartier divisor. Since X_Σ is normal, the group of Cartier divisors is naturally embedded into the group of Weil divisors.

Proposition 1.3.2. *Let $[D_h]$ be the Weil divisor associated to a Cartier divisor D_h. Then we have*

$$[D_h] = -\sum_{i=1}^{d} h(v_i)D_i.$$

From this we get a criterion for a Weil divisor $\sum a_i D_i$ to be Cartier: for each cone $\sigma \in \Sigma$ there exists $u(\sigma) \in M$ such that $\langle u(\sigma), v_i \rangle = -a_i$ whenever $\tau_i \subset \sigma$. Information about a Cartier divisor D_h can be read off from properties of the function h.

Proposition 1.3.3. *Let $D_h = \sum_{i=1}^{d} a_i D_i$ be a Cartier divisor. Then*
(i) D_h is generated by global sections if and only if $\langle u(\sigma), v_j \rangle \geq h(v_j) = -a_j$ whenever $\tau_j \not\subseteq \sigma$.
(ii) D_h is ample if and only if $\langle u(\sigma), v_j \rangle > h(v_j) = -a_j$ whenever $\tau_j \not\subseteq \sigma$ and σ is n-dimensional.

Those functions satisfying (i) are called *upper convex* and those satisfying (ii) are called *strictly upper convex.*
Since a toric variety X_Σ is Cohen-Macaulay, it has a dualizing sheaf ω_{X_Σ} and thus a canonical divisor K_{X_Σ}. Before finishing this section, we would like to determine the canonical divisor K_{X_Σ} on a toric variety X_Σ. This is given by the following:

Proposition 1.3.4. *The dualizing sheaf on a toric variety X_Σ is given by*

$$\omega_{X_\Sigma} = \mathcal{O}_X(-\sum_{i=1}^{d} D_i)$$

so that we have $K_{X_\Sigma} = -\sum_{i=1}^{d} D_i$.

1.4 Polarized Toric Varieties

First we define another type of combinatorial objects, namely polytopes:

Definition 1.4.1. *Let N, $N_{\mathbb{R}}$, M, $M_{\mathbb{R}}$ be as before.*

(i) A convex polytope $\Delta \subset M_{\mathbb{R}}$ is the convex hull of a finite number of points in $M_{\mathbb{R}}$. Δ is said to be integral if its vertices all lie in M. The dimension of Δ is defined to be the dimension of the subspace spanned by $\{u_1 - u_2 : u_1, u_2 \in \Delta\}$.

(ii) The polar(or dual) of $\Delta \subset M_{\mathbb{R}}$ is the set $\Delta^\circ \subset N_{\mathbb{R}}$ defined by

$$\Delta^\circ := \{v \in N_{\mathbb{R}} : \langle u, v\rangle \geq -1 \ for \ all \ u \in \Delta\},$$

which is also a convex polytope.

(iii) A face θ of Δ is a subset of the form

$$\theta = \{u \in \Delta : \langle u, v\rangle = r\}$$

for some $v \in N_{\mathbb{R}}$ and some $r \in \mathbb{R}$. A face of a convex polytope is also a convex polytope. As in the case of cones, a face is denoted as $\theta \prec \Delta$.

As in the case of cones, we have the following combinatorial result for polytopes:

Proposition 1.4.1. *If θ is a face of a convex polytope Δ, then*

$$\theta^\circ := \{v \in \Delta^\circ : \langle u, v\rangle = -1 \ for \ all \ u \in \theta\}$$

is a face of Δ°. This gives a one-to-one correspondence between the faces of Δ and the faces of Δ°. Also we have

$$dim(\theta) + dim(\theta^\circ) = dim(M_{\mathbb{R}}) - 1 = n - 1.$$

Now let $D_h = \sum_{i=1}^{d} a_i D_i$ be an ample T_N-invariant Cartier divisor on a toric variety X_Σ, i.e. we are given a polarized toric variety. Define

$$\Delta_h := \{u \in M_{\mathbb{R}} : \langle u, v_i\rangle \geq -a_i \ for \ all \ i\}.$$

Then Δ_h can be shown to be an integral convex polytope in $M_{\mathbb{R}}$ which contains 0 in its interior.

Conversely, given an integral convex polytope Δ in $M_{\mathbb{R}}$ which contains 0 in its interior, we can define a fan Σ_Δ in $N_{\mathbb{R}}$, called the *normal fan* of Δ, as follows. For an l-dimensional face $\theta \prec \Delta$, let $\sigma_\theta \subset N_{\mathbb{R}}$ be the dual of the l-dimensional cone

$$\check{\sigma_\theta} := \{\lambda(u - u') : u \in \Delta,\ u' \in \theta,\ \lambda \geq 0\} \subset M_{\mathbb{R}}.$$

We have $dim(\sigma_\theta) = n - dim(\theta)$ and we set $\Sigma_\Delta := \{\sigma_\theta : \theta \prec \Delta\}$. Then it can be shown that the normal fan defines a polarized toric variety: a projective toric variety we denote by $\mathbb{P}_\Delta$ together with an ample Cartier divisor D. In fact, if we consider monomials $t^k\chi^v$ where $v/k \in \Delta$, with multiplication defined by $t^k\chi^v \cdot t^l\chi^{v'} = t^{k+l}\chi^{v+v'}$, and let S_Δ be the $\mathbb{C}$-algebra generated by these monomials, then $\mathbb{P}_\Delta$ is given by:

$$\mathbb{P}_\Delta = \text{Proj } S_\Delta,$$

and the ample divisor is given by

$$D = \sum_{i=1}^{d} a_i D_i$$

where $a_i = -h(v_i) = -\inf\{\langle u, v_i\rangle : u \in M \cap \Delta\}$. Hence we get a one-to-one correspondence between integral convex polytopes and polarized toric varieties.

The preceding discussion can be illustrated again by the projective plane $X_\Sigma = \mathbb{P}^2$. It is compact since $|\Sigma| = N_{\mathbb{R}} = \mathbb{R}^2$. It is also smooth since each 2-dimensional cone in Σ is generated by a $\mathbb{Z}$-basis of N. The 1-dimensional cones are given by

$$\begin{aligned} \tau_0 &= \mathbb{R}_{\geq 0} v_0,\ v_0 = -e_1 - e_2, \\ \tau_1 &= \mathbb{R}_{\geq 0} v_1,\ v_1 = e_1, \\ \tau_2 &= \mathbb{R}_{\geq 0} v_2,\ v_2 = e_2. \end{aligned}$$

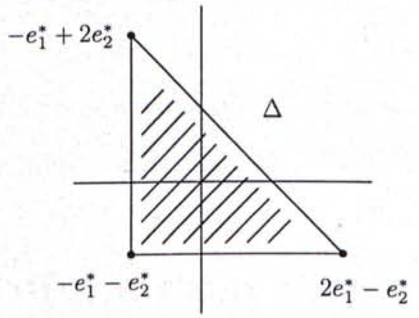
Denote the corresponding prime divisors by D_0, D_1 and D_2 respectively. Then the anticanonical divisor given by

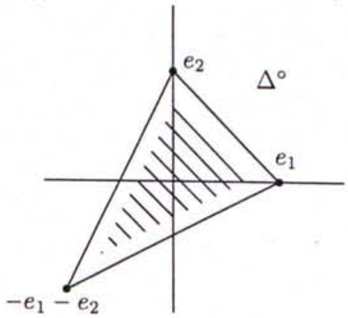
$$-K_{\mathbb{P}^2} = D_0 + D_1 + D_2$$

is ample. The polytope Δ corresponding to this polarization of $\mathbb{P}^2$ is then given by

$$\Delta = \{u \in M_{\mathbb{R}} = \mathbb{R}^2 : \langle u, v_i \rangle \geq -1 \text{ for } i = 1, 2, 3\},$$

which is the convex hull of the points $-e_1^* + 2e_2^*$, $2e_1^* - e_2^*$ and $-e_1^* - e_2^*$ in $M_{\mathbb{R}}$. This example can of course be generalized to n-dimensional projective space $\mathbb{P}^n$, $n \geq 1$.





Chapter 2

Calabi-Yau Manifolds from Toric Varieties

In this chapter we construct Calabi-Yau manifolds as hypersurfaces and complete intersections in toric Fano varieties. We follow the approaches in [2] and [3].

2.1 Toric Fano Varieties

We begin with the definition of toric Fano varieties:

Definition 2.1.1. *A compact toric variety X_Σ is Fano if its anticanonical divisor $-K_{X_\Sigma} = \sum_{i=1}^{d} D_i$ is Cartier and ample.*

Remark 2.1.1. *(i) In general the canonical divisor K_X of a Cohen-Macaulay variety X is Cartier if and only if X is Gorenstein, i.e. all of its local rings are Gorenstein. Hence the definition implies that a toric Fano variety is Gorenstein.*
(ii) By Proposition 1.3.3(ii), a toric variety X_Σ is Fano if and only if there is a strictly upper convex integral support function $h : N_\mathbb{R} \to \mathbb{R}$ with $h(v_i) = -1$ for $i = 1, ..., d$.

Since a toric Fano variety X_Σ is polarized by its anticanonical divisor, it is defined by an integral convex polytope $\Delta \subset M_\mathbb{R}$ with 0 in its interior: $X_\Sigma = \mathbb{P}_\Delta$.

A natural question then arises: how to characterize Δ when $\mathbb{P}_\Delta$ is Fano? In fact, it is the ingenious idea of Batyrev to introduce the notion of *reflexive polytopes* to answer this question.

Definition 2.1.2. *An integral convex polytope $\Delta \subset M_\mathbb{R}$ with 0 in its interior is called reflexive if its polar $\Delta^\circ \subset N_\mathbb{R}$ is also integral and contains 0 in its interior.*

Since $0 \in \text{Int}(\Delta)$ and $(\Delta^\circ)^\circ = \Delta$, Δ is reflexive if and only if Δ° is reflexive. We also have the following equivalent definition of reflexive polytopes:

Lemma 2.1.1. *$\Delta \subset M_\mathbb{R}$ is reflexive if and only if it satisfies the following (i) and (ii):*

(i) each codimension one face θ of Δ is supported by an affine hyperplane of the form $\{u \in M_\mathbb{R} : \langle u, v_\theta \rangle = -1\}$ for some $v_\theta \in N$.

(ii) $Int(\Delta) \cap M = \{0\}$.

Proof. This follows from the definition of the polar of a polytope and Proposition 1.4.1. □

Batyrev shows the following [2]:

Proposition 2.1.1. *Let $\mathbb{P}_\Delta$ be a projective toric variety polarized by the ample Cartier divisor D_h. Then $\mathbb{P}_\Delta$ is Fano if and only if Δ is reflexive.*

Proof. First suppose that $\mathbb{P}_\Delta$ is Fano. Then there exists a strictly upper convex integral support function $h : N_\mathbb{R} \to \mathbb{R}$ such that $h(v) = -1$ for each generator v of a 1-dimensional cone of the normal fan Σ_Δ of Δ. But recall that the normal fan consists of cones over the faces of the polar Δ° of Δ, so the 1-dimensional cones are cones over the vertices of Δ°. Hence the function h takes the value -1 on each vertex of Δ°. Such a vertex is of the form λv where $\lambda > 0$ and $v \in N$. But $h(\lambda v) = -1$ forces $\lambda = 1$ since $h(v) \in \mathbb{Z}$. This shows that Δ° is integral.
Conversely, suppose Δ is reflexive. Write $D_h = \sum_{i=1}^d a_i D_i$. Recall that

$$a_i = -h(v_i) = -\inf\{\langle u, v_i \rangle : u \in M \cap \Delta\}.$$

But Δ is reflexive, so $v_i \in N$ and

$$\langle u, v_i \rangle \geq -1 \text{ for all } u \in M \cap \Delta.$$

Hence $a_i = -(-1) = 1$ for all i. In other words, $D_h = \sum_{i=1}^{d} D_i = -K_{\mathbb{P}_\Delta}$ and $-K_{\mathbb{P}_\Delta}$ is thus Cartier and ample. By definition, $\mathbb{P}_\Delta$ is Fano. □

This proposition shows that toric Fano varieties are in one-to-one correspondence with reflexive polytopes.

The following are some examples of toric Fano varieties.

Example 2.1.1 We have already seen the example of $\mathbb{P}^2$. Now we generalize that to arbitrary dimensions. Let $N = \mathbb{Z}^n$ and $\{e_1, ..., e_n\}$ be the standard basis of N and let $e_0 := -\sum_{i=1}^{n} e_i$. Then the cones generated by subsets of $\{e_0, e_1, ..., e_n\}$ form a fan Σ in $N_\mathbb{R}$, which gives the projective space $X_\Sigma = \mathbb{P}^n$. On the other hand, let Δ be the convex hull of the $n+1$ points in $M_\mathbb{R} = \mathbb{R}^n$:

$$-\sum_{i=1}^{n} e_i^*, \ \ ne_j^* - \sum_{i \neq j} e_i^* \ for\ j = 1, ..., n.$$

Then Δ is a reflexive polytope with polar Δ° given by the convex hull of $\{e_0, e_1, ..., e_n\}$ in $N_\mathbb{R}$. Hence $\mathbb{P}^n = \mathbb{P}_\Delta$ is a toric Fano variety.

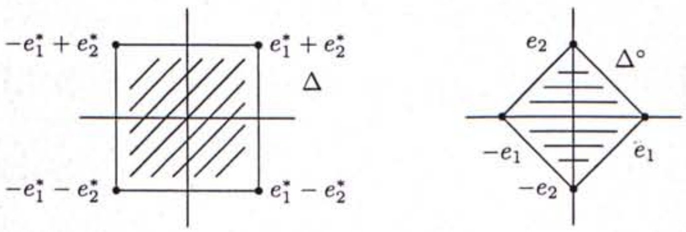
Example 2.1.2 Let n_1, n_2 be positive integers and denote $n := n_1 + n_2$. Let $N = \mathbb{Z}^n$ and $\{e_1, ..., e_{n_1}, f_1, ..., f_{n_2}\}$ be the standard basis of N and let $e_0 := -\sum_{i=1}^{n_1} e_i$, $f_0 := -\sum_{j=1}^{n_2} f_j$. Then the cones generated by subsets of $\{e_0, e_1, ..., e_{n_1}, f_0, f_1, ..., f_{n_2}\}$ form a fan Σ in $N_\mathbb{R}$, which gives the product of projective spaces $X_\Sigma = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$. On the other hand, let Δ be the convex hull of the $(n_1+1) \times (n_2+1)$ points in $M_\mathbb{R}$:

$$\Big\{-\sum_{i=1}^{n_1} e_i^*,\ n_1 e_j^* - \sum_{i \neq j} e_i^*,\ j = 1, ..., n_1\Big\} \bigoplus \Big\{-\sum_{i=1}^{n_2} f_i^*,\ n_2 f_j^* - \sum_{i \neq j} f_i^*,\ j = 1, ..., n_2\Big\}.$$

Then Δ is a reflexive polytope with polar Δ° given by the convex hull of the points $\{e_0, e_1, ..., e_{n_1}, f_0, f_1, ..., f_{n_2}\}$ in $N_\mathbb{R}$. Hence $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} = \mathbb{P}_\Delta$ is a toric Fano variety. A concrete example is given by $\mathbb{P}^1 \times \mathbb{P}^1$. The corresponding polytope Δ and its polar Δ° are the convex hulls of the points

$$\{e_1^* + e_2^*, -e_1^* + e_2^*, e_1^* - e_2^*, -e_1^* - e_2^*\} \text{ and } \{e_1, -e_1, e_2, -e_2\}$$

in $M_\mathbb{R}$ and $N_\mathbb{R}$ respectively.



More generally, one can construct $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ as a toric Fano variety. Also note that this is a special case of the construction in Remark 1.1.1(iii)

Example 2.1.3 Consider $\mathbb{Z}^{n+1}$ and its standard basis $\{e_0, e_1, ..., e_n\}$. Let $d_0, ..., d_n$ be positive integers and denote $d := \sum_{i=0}^n d_i$. Let N be the rank n lattice defined by

$$N := \mathbb{Z}^{n+1}/(\mathbb{Z} \cdot (d_0, \ldots, d_n))$$

Let v_i, $i = 0, 1, \ldots, n$ be the image of e_i in N. Then the cones generated by $\{v_0, v_1, \ldots, v_n\}$ form a fan Σ in $N_\mathbb{R}$, which gives the weighted projective space $X_\Sigma = \mathbb{P}(d_0, \ldots, d_n)$. In fact, $\mathbb{P}(d_0, \ldots, d_n) = (\mathbb{C}^{n+1} - \{0\})/\mathbb{C}^*$ where $\mathbb{C}^*$ acts by $\varsigma \cdot (z_0, \ldots, z_n) = (\varsigma^{d_0} z_0, \ldots, \varsigma^{d_n} z_n)$. However *not* all weighted spaces are Fano. The necessary and sufficient condition is given by the following:

Lemma 2.1.2. *$X_\Sigma = \mathbb{P}(d_0, ..., d_n)$ is Fano if and only if $d_i | d$ for $i = 0, 1, ..., n$.*

Proof. First note that the cone generators $v_o, v_1, ..., v_n$ satisfy $\sum_{i=0}^n d_i v_i = 0$. Consider the divisor $D = \sum_{i=0}^n D_i$. For each $i = 0, 1, ..., n$, there exists a unique $u_i \in M \otimes \mathbb{Q}$ such that $\langle u_i, v_j \rangle = -1$ for all $j \neq i$. Then D is Cartier if and only if $u_i \in M$ for all i. But

$$\langle u_i, v_i \rangle = \frac{\sum_{j \neq i} d_j}{d_i} = \frac{d}{d_i} - 1.$$

It follows that $u_i \in M$ if and only if $d_i | d$ for all i. Once D is Cartier, it is automatically ample since $\langle u_i, v_i \rangle > -1$. □

So for example, $\mathbb{P}(1,1,1,1,4)$, $\mathbb{P}(1,1,2,2,6)$ and $\mathbb{P}(1,1,1,6,9)$ are Fano, but $\mathbb{P}(1,1,2,3,5)$ and $\mathbb{P}(1,1,1,3,7)$ are not.

2.2 Calabi-Yau Hypersurfaces in Toric Fano Varieties

Let $\mathbb{P}_\Delta$ be an n-dimensional toric Fano variety associated to a reflexive polytope $\Delta \subset M_\mathbb{R}$ where rank $M = n$. Let $Z_f \in |-K_{\mathbb{P}_\Delta}|$ be a generic anticanonical hypersurface determined by $f \in H^0(-K_{\mathbb{P}_\Delta})$. Then

Proposition 2.2.1. *Z_f is a Calabi-Yau variety, i.e. its dualizing sheaf ω_{Z_f} is trivial, and $H^k(Z_f, \mathcal{O}_f) = 0$ for $0 < k < n-1$.*

Proof. By Remark 2.1.1(i), $\mathbb{P}_\Delta$ is Gorenstein, and hence has at most canonical singularities [2]. By a theorem of Bertini type [28], we know that Z_f also has at most canonical singularities. Now since $\mathbb{P}_\Delta$ is Cohen-Macaulay and $-K_{\mathbb{P}_\Delta}$ is Cartier, we can use the adjunction formula to give

$$\omega_{Z_f} \cong \omega_{\mathbb{P}_\Delta}(-K_{\mathbb{P}_\Delta})|_{Z_f} \cong \mathcal{O}_{Z_f}.$$

On the other hand, as $\mathcal{O}_{\mathbb{P}_\Delta}(-Z_f) \cong \mathcal{O}_{\mathbb{P}_\Delta}(K_{\mathbb{P}_\Delta}) = \omega_{\mathbb{P}_\Delta}$ we have

$$0 \to \omega_{\mathbb{P}_\Delta} \to \mathcal{O}_{\mathbb{P}_\Delta} \to \mathcal{O}_{Z_f} \to 0$$

which induces the long exact sequence

$$\ldots \to H^k(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) \to H^k(Z_f, \mathcal{O}_{Z_f}) \to H^{k+1}(\mathbb{P}_\Delta, \omega_{\mathbb{P}_\Delta}) \to \ldots.$$

But for a compact toric variety X_Σ, we always have $H^q(X_\Sigma, \mathcal{O}_{X_\Sigma}) = 0$ for $q > 0$. So $H^k(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}) = 0$ for $k > 0$. And by Serre duality, $H^{k+1}(\mathbb{P}_\Delta, \omega_{\mathbb{P}_\Delta}) \cong$

$H^{n-k-1}(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta})^\vee = 0$ for $k < n-1$. Altogether we get $H^k(Z_f, \mathcal{O}_{Z_f}) = 0$ for $0 < k < n-1$. □

In fact if we take a generic anticanonical hypersurface in any Fano variety (i.e. not necessarily toric), we still get a Calabi-Yau variety. But the point in considering just the toric case is that, as we can see below, the singular Calabi-Yau Z_f above can always be resolved into a Calabi-Yau orbifold with at most terminal singularities and with (some of) its Hodge numbers given in explicit formulae.

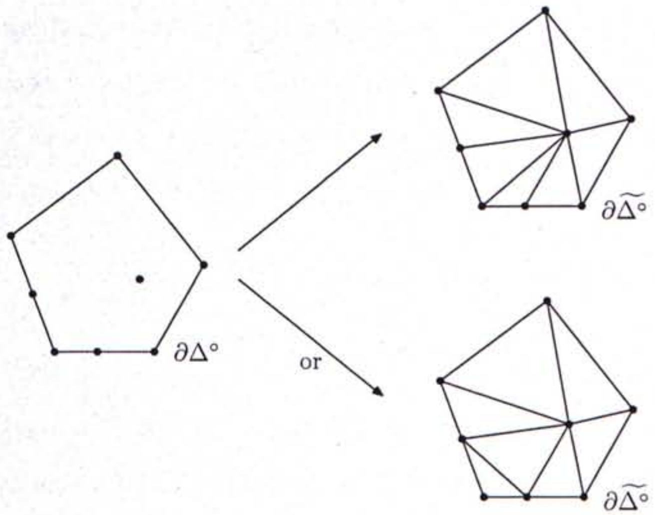
Of course, if the toric Fano variety is already smooth then the generic anticanonical hypersurfaces give examples of smooth Calabi-Yau manifolds. This is the case when, for example, $\mathbb{P}_\Delta = \mathbb{P}^4$. The corresponding Calabi-Yau 3-fold is the well-known quintic 3-fold. More generally, a generic degree $n+1$ hypersurface in $\mathbb{P}^n$ is a smooth Calabi-Yau $(n-1)$-fold. Other examples are given by generic hypersurfaces of appropriate multi-degree in product of projective spaces, e.g. the generic hypersurface of multi-degree $(3,2,2)$ in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a smooth Calabi-Yau 3-fold.

Returning to the general situation when the toric Fano variety is not smooth, we shall construct a partial resolution of Z_f which is *minimal* in a certain sense and the resolved variety will be a *Calabi-Yau orbifold with terminal singularities.* In particular, this resolved variety will be smooth in codimension three. Hence when $dim(Z_f) = n-1 = 3$, we still get a *smooth* Calabi-Yau 3-fold even if the ambient space is singular.

To begin with, recall that the normal fan $\Sigma_\Delta \subset N_\mathbb{R}$ of $\Delta \subset M_\mathbb{R}$ consists of cones over faces of the polar $\Delta^\circ \subset N_\mathbb{R}$. Then let $\widetilde{\Delta^\circ} \subset N_\mathbb{R}$ be the subdivision of Δ° satisfying the followings:

(i) The vertices of $\widetilde{\Delta^\circ}$ are precisely the points in $\partial\Delta^\circ \cap N$; and

(ii) Any k-dimensional face of $\widetilde{\Delta^\circ}$ is the convex hull of $k+1$ lattice points in N (in such a way that besides these $k+1$ points, the convex hull contains no other lattice points). Such subdivisions exist by a result in [26] and there are more than one choice in general:



Define Σ to be the set of all cones over faces of $\widetilde{\Delta^\circ}$. If we denote by $\varphi : N \to N$ the identity, then it follows from Corollary 1.2.2 that the induced morphism $\tau := \varphi_* : X_\Sigma \to X_{\Sigma_\Delta} = \mathbb{P}_\Delta$ is birational and proper.

Proposition 2.2.2. *The partial resolution $\tau : X_\Sigma \to \mathbb{P}_\Delta$ is crepant (or minimal), i.e.*

$$\tau^*(K_{\mathbb{P}_\Delta}) = K_{X_\Sigma}.$$

Proof. First note that for each $i = 1, ..., d$ the pullback $\tau^*(D_i)$ of a prime divisor D_i in $\mathbb{P}_\Delta$ remains a prime divisor in X_Σ since τ is a toric blow-up. Thus if h is the support function that corresponds to the canonical divisor $K_{\mathbb{P}_\Delta}$ of $\mathbb{P}_\Delta$, then h also corresponds to the divisor $\tau^*(K_{\mathbb{P}_\Delta})$. Now it follows from the proof of Proposition 2.1.1 that h equals -1 on $\partial\Delta^\circ$. But the generators of the 1-dimensional cones of Σ are all contained in $\partial\Delta^\circ$. Hence h equals -1 on all the generators of the 1-dimensional cones of Σ. This says exactly that h corresponds to the canonical divisor K_{X_Σ} of X_Σ. The result now follows from Proposition 1.3.1. □

Remark 2.2.1. *(The proof of) the proposition also shows that K_{X_Σ} (or $-K_{X_\Sigma}$) is Cartier and generated by global sections.*

Before stating the next proposition, let's review some facts about canonical and terminal singularities. Recall that a normal and quasiprojective variety X is said to have *canonical singularities* if it satisifies:

(i) for some integer $r \geq 1$, rK_X is Cartier, and

(ii) if $f : Y \to X$ is a resolution of singularities and $\{E_i\}$ is the family of exceptional prime divisors of f, then

$$rK_Y = f^*(rK_X) + \sum a_i E_i$$

where $a_i \geq 0$ for all i.

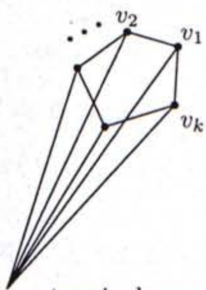
If further we have $a_i > 0$ for all i, then X is said to have *terminal singularities.* Now for toric varieties, we have the following definition of Reid:

Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex cone and let $\{v_1, \ldots, v_k\}$ be the generators of the 1-dimensional faces of σ. We say that σ is *terminal* (resp. *canonical*) if it satisifies (i) and (ii) (resp. (i) and (ii')) below:

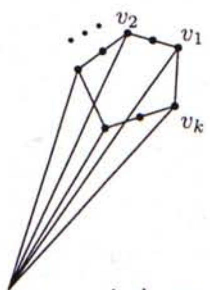
(i) $v_1, \ldots, v_k$ are contained in an affine hyperplane $H = \{v \in N_{\mathbb{R}} : \langle u, v\rangle = -1\}$ for some $u \in M$;

(ii) $N \cap \sigma \cap \{v \in N_{\mathbb{R}} : \langle u, v\rangle \geq -1\} = \{0, v_1, \ldots, v_k\}$.

((ii') $N \cap \sigma \cap \{v \in N_{\mathbb{R}} : \langle u, v\rangle > -1\} = \{0\}$.)



a terminal cone



a canonical cone

Reid's criterion (cf. (1.12) of [29]) then says that a toric variety X_Σ has terminal (resp. canonical) singularities if and only if $\forall \sigma \in \Sigma$, σ is terminal (resp. canonical).

Proposition 2.2.3. *The resolved variety X_Σ has terminal singularities.*

Proof. Let $\sigma \in \Sigma$. By definition, the generators $\{v_1, ..., v_k\}$ of the 1-dimensional faces of σ are contained in an affine hyperplane $H = \{v \in N_{\mathbb{R}} : \langle u, v \rangle = -1\} \subset N_{\mathbb{R}}$ for some $u \in M$; and there are no other lattice points of N in the part of σ under or on H, i.e. $N \cap \sigma \cap \{v \in N_{\mathbb{R}} : \langle u, v \rangle \leq -1\} = \{0, v_1, ..., v_k\}$. By Reid's criterion, X_Σ has terminal singularities. □

It also follows from the definition that Σ is simplicial, i.e. each cone $\sigma \in \Sigma$ is generated by $\mathbb{R}$-linearly independent vectors in $N_{\mathbb{R}}$. So X_Σ is an orbifold by Proposition 1.2.2. Together with the above proposition, this implies

Corollary 2.2.1. *X_Σ is smooth in codimension three.*

Now let $\widehat{Z_f} := \tau^{-1}(Z_f)$ be the proper transform of Z_f via τ, together with the partial resolution

$$\tau : \widehat{Z_f} \to Z_f.$$

Proposition 2.2.2 shows that $\widehat{Z_f}$ is a generic anticanonical hypersurface in X_Σ, and hence an orbifold with terminal singularities; and $\tau : \widehat{Z_f} \to Z_f$ is crepant. Also, by (the proof of) Proposition 2.2.1, $\widehat{Z_f}$ is a Calabi-Yau variety.

Remark 2.2.2. *(i) In [2] Batyrev termed the morphisms $\tau : X_\Sigma \to \mathbb{P}_\Delta$ and $\tau : \widehat{Z_f} \to Z_f$ as maximal projective crepant partial desingularizations (abbreviated as MPCP-desingularizations).*
(ii) In the sense of the Mori Program, $\widehat{Z_f}$ is a minimal model of Z_f.

Let's look at a concrete example of MPCP-desingularizations. Take $N = \mathbb{Z}^4$ with its standard basis $\{e_1, e_2, e_3, e_4\}$ and let Σ be the fan in $N_{\mathbb{R}} = \mathbb{R}^4$ generated by $\{v_0, v_1, v_2, v_3, v_4\}$ where $v_0 = -e_1 - 2e_2 - 2e_3 - 6e_4$ and $v_i = e_i$ for $i = 1, 2, 3, 4$. Then

$$v_0 + v_1 + 2v_2 + 2v_3 + 6v_4 = 0.$$

Hence the toric variety corresponding to Σ is the 4-dimensional weighted projective space $\mathbb{P}(1,1,2,2,6)$. By Lemma 2.1.2, this is a toric Fano variety. Denote by Δ the corresponding reflexive polytope. Then the vertices of the polar $\Delta^\circ \subset N_\mathbb{R}$ are exactly the points $\{v_0, v_1, v_2, v_3, v_4\}$, and Δ is the convex hull of the points

$$\begin{aligned} u_0 &= (-1,-1,-1,-1), \\ u_1 &= (11,-1,-1,-1), \\ u_2 &= (-1,5,-1,-1), \\ u_3 &= (-1,-1,5,-1), \\ u_4 &= (-1,-1,-1,5). \end{aligned}$$

If $(z_0, z_1, z_2, z_3, z_4)$ are weighted homogeneous coordinates for $\mathbb{P}(1,1,2,2,6)$, then the singularities is along the surface $z_0 = z_1 = 0$ which corresponds to the cone generated by v_0 and v_1 (or the codimension 2 face $\theta^\circ \prec \Delta^\circ$ with vertices v_2, v_3, v_4). Let $v_5 := \frac{1}{2}(v_0+v_1) = (0,-1,-1,3)$. Then $\partial\Delta^\circ \cap N = \{v_0, v_1, v_2, v_3, v_4, v_5\}$. Hence the MPCP-desingularization of $\mathbb{P}(1,1,2,2,6)$ is obtained by inserting the vertex v_5 into Δ° and subdividing the polytope. In geometric terms, this corresponds to blowing up the surface $z_0 = z_1 = 0$. In this example, the choice of subdivision is unique. But in general, there are several different ways in subdividing the polytope and the resulting Calabi-Yau orbifolds differ by so-called *flops*.

2.3 Computation of Hodge Numbers of $\widehat{Z_f}$

By general Hodge theory, generic hypersurfaces of an orbifold have *pure Hodge structures* on their cohomology groups. In particular it is the case for $\widehat{Z_f}$. Thus it makes sense to speak of the Hodge numbers of Z_f. In this section, we shall show how to calculate $h^{n-2,1}(\widehat{Z_f})$ and $h^{1,1}(\widehat{Z_f})$, expressing them in terms of the combinatorial data of Δ.

2.3.1 The results of Danilov and Khovanskiĭ

In order to compute $h^{n-2,1}(\widehat{Z_f})$, we need to recall some results from [15]. By Deligne's Hodge theory [16] for any complex algebraic variety X (even noncompact or singular), the cohomology groups with compact support $H^k_c(X)$ are endowed with the so-called *mixed Hodge structures*. The *Hodge-Deligne numbers* are then defined as $h^{p,q}(H^k_c(X)) := h^{p,q}(Gr^W_{p+q}(H^k_c(X)))$. Danilov and Khovanskiĭ further introduced the following invariant of X in [15]:

$$e^{p,q}(X) := \sum_k (-1)^k h^{p,q}(H^k_c(X)).$$

Observe that $e^{p,q}(X) = e^{q,p}(X)$. If X is an orbifold, then $H^k_c(X) = H^k(X)$ actually carries pure Hodge structures, so $h^{p,q}(H^k(X)) = 0$ if $p+q \neq k$ and we have $e^{p,q}(X) = (-1)^{p+q}h^{p,q}(H^{p+q}(X)) = (-1)^{p+q}h^{p,q}(X)$. In addition, the followings were proved in [15]:

Proposition 2.3.1. *(i) Suppose X is a disjoint union of a finite number of locally closed subvarieties X_i for $i \in I$. Then*

$$e^{p,q}(X) = \sum_{i \in I} e^{p,q}(X_i).$$

(ii) If $X = X' \times X''$ is a product of two complex algebraic varieties, then

$$e^{p,q}(X) = \sum_{(p'+p'',q'+q'')=(p,q)} e^{p',q'}(X') \cdot e^{p'',q''}(X'').$$

(i) is proved by using the exact sequence of Hodge structures:

$$\ldots \to H^k_c(X \backslash Y) \to H^k_c(X) \to H^k_c(Y) \to H^{k+1}_c(X \backslash Y) \to \ldots$$

for a closed subvariety Y in X. (ii) follows from the *Künneth isomorphism* $H^*_c(X) \otimes H^*_c(Y) \to H^*_c(X \times Y)$ which is compatible with the Hodge structures.

Now let $\mathbb{P}_\Delta$ be a polarized toric variety associated with an integral convex polytope $\Delta \subset M_{\mathbb{R}}$. Let $Z_0 \subset U_{\{0\}} \cong (\mathbb{C}^*)^n$ be a hypersurface obtained by restricting a divisor Z of $\mathbb{P}_\Delta$ with $Z \in |\mathcal{O}_{\mathbb{P}_\Delta}(1)|$.

Definition 2.3.1. *We say that Z_0 is Δ-regular if Z has a transversal intersection with all the strata of $\mathbb{P}_\Delta$, i.e. it has smooth intersection with all the orbits O_σ of $\mathbb{P}_\Delta$.*

Making use of Proposition 2.3.1 and other results, Danilov and Khovanskiĭ obtained some general formulae for the Hodge numbers $h^{n-2,1}(H_c^{n-1}(Z_0))$ and $h^{n-2,0}(H_c^{n-1}(Z_0))$ which we need in computing $h^{n-2,1}(\widehat{Z_f})$. Before stating their results, we have to fix some notations:

Definition 2.3.2. *We denote by $l(\Delta)$ the number of integral points in Δ and by $l^*(\Delta)$ the number of integral points in the interior of Δ.*

Proposition 2.3.2. *Let $dim(\Delta) = n \geq 4$. Then*
(i) $h^{n-2,1}(H_c^{n-1}(Z_0)) + h^{n-2,0}(H_c^{n-1}(Z_0)) = l^*(2\Delta) - (n+1)l^*(\Delta)$;
(ii) $h^{n-1,0}(H_c^{n-1}(Z_0)) = l^*(\Delta)$;
(iii) $h^{n-2,0}(H_c^{n-1}(Z_0)) = \sum_{codim(\theta)=1} l^*(\theta)$
where θ ranges over faces of codimension one of Δ.

Remark 2.3.1. *When Δ is a reflexive polytope, its codimension 1 faces are given by $\{u \in \Delta : \langle u, v\rangle = -1\}$ where v is a vertex of Δ° which is integral since Δ° has integral vertices. But the integral points u in the interior of 2Δ satisfy $\langle u, v\rangle > -2$ for all vertices v of Δ°, and hence $\langle u, v\rangle \geq -1$. It follows that $l^*(2\Delta) = l(\Delta)$. Similarly, $l^*(\Delta) = 1$.*

Danilov and Khovanskiĭ also proved a theorem of Lefschetz type, which will also be used in the sequel:

Theorem 2.3.1. *For a Δ-regular hypersurface $Z_0 \subset (\mathbb{C}^*)^n$, we have*
(i) $h^{p,q}(H_c^k(Z_0)) = 0$ *for* $k < n-1$;
(ii) $h^{p,q}(H_c^{n-1}(Z_0)) = 0$ *for* $p+q > n-1$;
(iii) $h^{p,q}(H_c^k(Z_0)) = h^{p,q}(H_c^{k+2}((\mathbb{C}^*)^n))$ *for* $k > n-1$.

The Hodge numbers of the torus $(\mathbb{C}^*)^n$ is given by

$$h^{p,q}(H^k_c((\mathbb{C}^*)^n)) = \begin{cases} C^p_n & \text{if } p = q \text{ and } k = n + p, \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2 The Hodge number $h^{n-2,1}(\widehat{Z_f})$

We are now ready to compute $h^{n-2,1}(\widehat{Z_f})$.

Theorem 2.3.2. *For $n \geq 4$ the Hodge number $h^{n-2,1}(\widehat{Z_f})$ is given by*

$$h^{n-2,1}(\widehat{Z_f}) = l(\Delta) - n - 1 - \sum_{codim(\theta)=1} l^*(\theta) + \sum_{codim(\theta)=2} l^*(\theta)l^*(\theta^\circ)$$

where θ in the summations denote faces of Δ and θ° denote the face in Δ° dual to θ.

Proof. First note that since Z_f is quasi-smooth and compact, we have $h^{n-2,1}(\widehat{Z_f}) = (-1)^{n-1}e^{n-2,1}(\widehat{Z_f})$. By Proposition 2.3.1(i), the key is then to find a 'good' stratification of $\widehat{Z_f}$. Recall that we have the proper birational morphism $\tau : \widehat{Z_f} \to Z_f$. Through this map, $\widehat{Z_f}$ can be represented as a disjoint union:

$$\widehat{Z_f} = \coprod_{\theta \prec \Delta} \tau^{-1}(Z_{f,\theta})$$

where $Z_{f,\theta} = Z_f \cap O_{\sigma_\theta}$ and σ_θ is the cone in $N_\mathbb{R}$ over the dual face $\theta^\circ \prec \Delta^\circ$. On the other hand, all irreducible components of fibers of τ over closed points of $Z_{f,\theta}$ are toric varieties. A stratification of $\tau^{-1}(Z_{f,\theta})$ is thus given by smooth affine varieties isomorphic to $Z_{f,\theta} \times (\mathbb{C}^*)^k$, $k \geq 0$. Therefore we obtain a stratification of $\widehat{Z_f}$ by smooth affine varieties which are isomorphic to $Z_{f,\theta} \times (\mathbb{C}^*)^k$, $\theta \prec \Delta$, $k \geq 0$. Next by Proposition 2.3.1(ii), Theorem 2.3.1 and the formulae for the Hodge numbers of the torus, we can conclude that $e^{n-2,1}(Z_{f,\theta} \times (\mathbb{C}^*)^k)$ is nonzero only if $\theta = \Delta$ and $k = 0$; or $dim(\theta) = n-2$ and $k = 1$. In the first case: $\theta = \Delta$, $Z_{f,\Delta} = Z_f \cap O_{\{0\}} = Z_0$ is just the affine hypersurface in $U_{\{0\}} \cong (\mathbb{C}^*)^n$ obtained by restricting the divisor Z_f. By Proposition 2.3.2(i) and (iii), we have

$$\begin{aligned} e^{n-2,1}(Z_0) &= (-1)^{n-1}(l^*(2\Delta) - (n+1)l^*(\Delta) - \sum_{codim(\theta)=1} l^*(\theta)) \\ &= (-1)^{n-1}(l(\Delta) - (n+1) - \sum_{codim(\theta)=1} l^*(\theta)). \end{aligned}$$

For the latter case: $dim(\theta) = n-2$ and $k = 1$. The strata which are isomorphic to $Z_{f,\theta} \times \mathbb{C}^*$ appear in the fibers of τ over $(n-3)$-dimensional singular affine subvarieties $Z_{f,\theta} \subset Z_f$ having codimension 2 in Z_f. By the very definition of the subdivision $\widetilde{\Delta^\circ}$ of Δ°, τ is $l^*(\theta^\circ)$ to 1 on $\tau^{-1}(Z_{f,\theta})$ (Note that $dim(\theta) = n-2$ implies $dim(\theta^\circ) = 1$ and the number of newly-added vertices to θ° is equal to $l^*(\theta^\circ)$). In other words, $\tau^{-1}(Z_{f,\theta})$ has $l^*(\theta^\circ)$ strata isomorphic to $Z_{f,\theta} \times \mathbb{C}^*$ for every codimension 2 face θ of Δ. Now by Proposition 2.3.1(ii), Proposition 2.3.2(ii) and the formulae for the Hodge numbers of the torus, we have

$$e^{n-2,1}(Z_{f,\theta} \times \mathbb{C}^*) = e^{n-3,0}(Z_{f,\theta}) \cdot e^{1,1}(\mathbb{C}^*) = (-1)^{n-3}h^{n-3,0}(Z_{f,\theta}) = (-1)^{n-1}l^*(\theta).$$

Hence the codimension 2 faces contribute

$$(-1)^{n-1} \sum_{codim(\theta)=2} l^*(\theta)l^*(\theta^\circ)$$

to $e^{n-2,1}(\widehat{Z_f})$. Therefore altogether we get

$$\begin{aligned} h^{n-2,1}(\widehat{Z_f}) &= (-1)^{n-1}e^{n-2,1}(Z_f) \\ &= (-1)^{n-1}(e^{n-2,1}(Z_0) + \sum_{codim(\theta)=2} l^*(\theta^\circ)e^{n-2,1}(Z_{f,\theta} \times \mathbb{C}^*)) \\ &= l(\Delta) - n - 1 - \sum_{codim(\theta)=1} l^*(\theta) + \sum_{codim(\theta)=2} l^*(\theta)l^*(\theta^\circ). \end{aligned}$$

□

2.3.3 The Hodge number $h^{1,1}(\widehat{Z_f})$

Theorem 2.3.3. *For $n \geq 4$ the Hodge number $h^{1,1}(\widehat{Z_f})$ is given by*

$$h^{1,1}(\widehat{Z_f}) = l(\Delta^\circ) - 1 - n - \sum_{codim(\theta^\circ)=1} l^*(\theta^\circ) + \sum_{codim(\theta^\circ)=2} l^*(\theta^\circ)l^*(\theta)$$

where $\theta \prec \Delta$ and $\theta^\circ \prec \Delta^\circ$ are dual faces.

Proof. Since $n \geq 4$, $h^{2,0}(\widehat{Z_f}) = 0$. Hence rk $\mathrm{Pic}_{\mathbb{Q}}(\widehat{Z_f}) = h^{1,1}(\widehat{Z_f})$. $\mathrm{Pic}_{\mathbb{Q}}(\widehat{Z_f})$ is generated by the classes of components of $\widehat{Z_f} - \widehat{Z_{f,0}}$ where $\widehat{Z_{f,0}} = \widehat{Z_f} \cap (\mathbb{C}^*)^n$ is the affine part of $\widehat{Z_f}$. On the other hand, the classes of these components are *not* independent. In fact, the group they generate contains the restrictions of T_N-invariant divisors of X_Σ on $\widehat{Z_f}$. There are n relations on $\mathrm{Pic}_{\mathbb{Q}}(\widehat{Z_f})$ given by the globally linear functions over N (defined by $m \in M$) which correspond to principal Cartier divisors. It can be proved that these are indeed all the relations among the components of $\widehat{Z_f} - \widehat{Z_{f,0}}$ [2]. Therefore,

$$h^{1,1}(\widehat{Z_f}) = \text{rk } \mathrm{Pic}_{\mathbb{Q}}(\widehat{Z_f}) = (\text{number of components of } \widehat{Z_f} - \widehat{Z_{f,0}}) - n.$$

The components of $\widehat{Z_f} - \widehat{Z_{f,0}}$ come from divisors D_i restricted on Z_f. Each v_i (the generator of $\tau_i \in \Sigma(1)$) lies in a face $\theta^\circ \prec \Delta^\circ$. Recall that as τ is a toric blowup, we have $\tau(D_i) = V_{\sigma_\theta}$. There are several cases:

Case 1: $codim(\theta^\circ) = 1$. If v_i is an interior point of θ°, then $\tau(D_i)$ is a point and $D_i \cap \widehat{Z_f} = \emptyset$ for a general $\widehat{Z_f}$. This gives

$$|\Sigma(1)| - \sum_{codim(\theta^\circ)=1} l^*(\theta^\circ) = l(\Delta^\circ) - 1 - \sum_{codim(\theta^\circ)=1} l^*(\theta^\circ)$$

components. Note that we have used $|\Sigma(1)| = |\Delta^\circ \cap N - \{0\}| = l(\Delta^\circ) - 1$.

Case 2: $codim(\theta^\circ) = 2$. Then $dim(\theta) = codim(\theta^\circ) - 1 = 1$ and V_{σ_θ} is a curve. By the intersection theory of toric varieties [19],

$$Z_f \cdot V_{\sigma_\theta} = l^*(\theta) + 1,$$

i.e. $Z_f \cap V_{\sigma_\theta}$ has $l^*(\theta) + 1$ points, so that $D_i \cap \widehat{Z_f}$ has $l^*(\theta) + 1$ connected components. But their sum is already counted in case 1. Hence these totally gives, in addition,

$$\sum_{codim(\theta^\circ)=2} l^*(\theta^\circ) l^*(\theta)$$

components.

Case 3: $codim(\theta^\circ) \geq 3$. Then $dim(\theta) \geq 2$. By Bertini theorem, $Z_f \cap V_{\sigma_\theta}$ is irreducible. Thus these give no new components.

The result follows. □

As mentioned in the introduction, physicists discovered a duality, called the Mirror Symmetry for Calabi-Yau 3-folds. In fact, many of the early constructions of Calabi-Yau 3-folds are for the purpose of verifying Mirror Symmetry, out of which the family of quintic in $\mathbb{P}^4$ and its mirror is the most famous (cf. [6]). After the work [6], many examples of mirror pairs have been constructed. In 1993, Batyrev introduced (in [2]) the notion of reflexive polytopes and construct Calabi-Yau 3-folds as hypersurfaces in toric variety. This, on the one hand, greatly generalized the previous constructions of mirror pairs; and on the other hand, gave a possible mathematical interpretation of Mirror Symmetry. Notice that the formulae for $h^{n-2,1}(\widehat{Z_f})$ and $h^{1,1}(\widehat{Z_f})$ given in this section are interchanged when the roles of Δ and Δ° are interchanged. In view of this, Batyrev proposed that the families of Calabi-Yau 3-folds from dual polytopes Δ and Δ° form a mirror pair. He was able to show that his construction of mirror pairs generalized the previous ones. Details can be found in [2].

2.4 Calabi-Yau Complete Intersections in Toric Fano Varieties

We now come to the construction of Calabi-Yau manifolds as complete intersections in toric Fano varieties. To do this, we need to introduce the notion of a *nef-partition*. Let $\mathbb{P}_\Delta$ be an n-dimensional toric Fano variety associated to a reflexive polytope Δ. Let Σ_Δ be the normal fan of Δ. Given a partition

$$\Sigma_\Delta(1) = I_1 \cup \ldots \cup I_r$$

into r disjoint subsets, we define divisors E_j for $1 \leq j \leq r$ as:

$$E_j := \sum_{i \in I_j} D_i.$$

Then we have $-K_{\mathbb{P}_\Delta} = E_1 + \ldots + E_r$.

Definition 2.4.1. *The decomposition $\Sigma_\Delta(1) = I_1 \cup \ldots \cup I_r$ is called a nef-partition if for each j, E_j is a Cartier divisor generated by its global sections. (or equivalently, Δ is a Minkowski sum $\Delta_1 + \ldots + \Delta_r$ and $\Delta_i \cap \Delta_j = \{0\}$ for any $i \neq j$.)*

Let $f_j \in H^0(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(E_j))$ be generic global sections. Let $V \subset \mathbb{P}_\Delta$ be the complete intersection of dimension $n - r$ defined by $f_1 = \ldots = f_r = 0$. Recall that we have a *MPCP*-desingularization $\tau : X_\Sigma \to \mathbb{P}_\Delta$. Let $\widehat{V} := \tau^{-1}(V)$. Then $\widehat{V}$ is again a complete intersection in X_Σ defined by global sections of *semi-ample sheaves* (i.e. sheaves generated by global sections). By the adjunction formula, V has trivial canonical sheaf, and so does $\widehat{V}$ because $\tau : \widehat{V} \to V$ is crepant.

Lemma 2.4.1. *Let D be a divisor on a polarized toric variety $\mathbb{P}_\Delta$ such that $\mathcal{O}_{\mathbb{P}_\Delta}(D)$ is generated by global sections. Let Δ_D be the polytope associated to D. Then we have:*

$$H^i(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(-D)) = \begin{cases} 0 & \text{if } i \neq dim(\Delta_D) \\ l^*(\Delta_D) & \text{if } i = dim\Delta_D \end{cases}$$

Proof. Let $k := dim(\Delta_D)$. Then the invertible sheaf $\mathcal{O}_{\mathbb{P}_\Delta}(D)$ defines the canonical morphism:

$$\pi_D : \mathbb{P}_\Delta \to \mathbb{V} := \text{Proj} \bigoplus_{m \geq 0} H^0(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(mD))$$

where $\mathbb{V} \cong \mathbb{P}_{\Delta_D}$ is a k-dimensional polarized toric variety and $\mathcal{O}_{\mathbb{P}_\Delta}(D) \cong \pi_D^* \mathcal{O}_{\mathbb{V}}(1)$. Since π_D is finite, we have

$$H^i(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(-D)) \cong H^i(\mathbb{V}, \mathcal{O}_{\mathbb{V}}(-1)).$$

But $H^i(\mathbb{V}, \mathcal{O}_{\mathbb{V}}(-1)) = 0$ for $i < k$ and $H^k(\mathbb{V}, \mathcal{O}_{\mathbb{V}}(-1)) = l^*(\Delta_D)$ [14]. □

Proposition 2.4.1. *Let $\mathbb{P}_\Delta$ be an n-dimensional polarized toric variety and Δ_i for $1 \leq i \leq r$ be the supporting polytopes for global sections of some semi-ample invertible sheaves $\mathcal{L}_i$ on $\mathbb{P}_\Delta$ such that $dim(\Delta_1 + \ldots + \Delta_r) = n$. Let $f_i \in H^0(\mathbb{P}_\Delta, \mathcal{L}_i)$ be generic sections and let Z_i be the zero set of f_i. Let W be the complete intersection $Z_1 \cap \ldots \cap Z_r$. Assume that $n - r \geq 2$ and for any subset $\{\Delta_{i_1}, \ldots, \Delta_{i_s}\} \subset \{\Delta_1, \ldots, \Delta_r\}$,*

$$l^*(\Delta_{i_1} + \ldots + \Delta_{i_s}) = \begin{cases} 1 & \text{if } s = r; \\ 0 & \text{if } s < r. \end{cases}$$

Then we have

$$h^1(\mathcal{O}_W) = \ldots = h^{n-r-1}(\mathcal{O}_W) = 0 \text{ and } h^0(\mathcal{O}_W) = h^{n-r}(\mathcal{O}_W) = 1.$$

Proof. Denote by $\mathcal{K}^*$ the *Koszul complex*:

$$\mathcal{O}_{\mathbb{P}_\Delta}(-Z_1 - \ldots - Z_r) \to \ldots \to \sum_{i<j} \mathcal{O}_{\mathbb{P}_\Delta}(-Z_i - Z_j) \to \sum_i \mathcal{O}_{\mathbb{P}_\Delta}(-Z_i) \to \mathcal{O}_{\mathbb{P}_\Delta}.$$

There are two spectral sequences $'E$ and $''E$ converging to the *hypercohomology* $\mathbb{H}^*(\mathbb{P}_\Delta, \mathcal{K}^*)$:

$$'E_2^{p,q} = H^p(\mathbb{P}_\Delta, \mathcal{H}^q(\mathcal{K}^*)),$$

$$''E_2^{p,q} = H^q(\mathbb{P}_\Delta, H^p(\mathcal{K}^*)).$$

The fact that $\mathcal{K}^*$ is an acyclic resolution of $\mathcal{O}_W$ implies:

$$\mathcal{H}^q(\mathcal{K}^*) = \begin{cases} 0 & \text{for } q \neq r; \\ \mathcal{O}_W & \text{for } q = r. \end{cases}$$

It follows that $'E$ degenerates at $'E_2$ and we have

$$\mathbb{H}^{r+p}(\mathbb{P}_\Delta, \mathcal{K}^*) \cong H^p(\mathbb{P}_\Delta, \mathcal{O}_W) \cong H^p(W, \mathcal{O}_W).$$

Now $''E_2^{p,q}$ is the cohomology of the bicomplex:

$$''E_1^{p,q} = \bigoplus_{\{i_1,\ldots,i_{r-p}\}} H^q(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(-Z_{i_1} - \ldots - Z_{i_{r-p}})).$$

By the previous lemma and the assumptions we made, we have

$$''E_1^{r,0} \cong \; ''E_1^{0,n} \cong \mathbb{C}$$

and all other $''E_1^{p,q}$ equal to zero. Hence $''E$ degenerates at $''E_1$ and we have isomorphisms:

$$\mathbb{H}^n(\mathbb{P}_\Delta, \mathcal{K}^*) \cong \; ''E_1^{0,n} \cong \mathbb{C},$$

$$\mathbb{H}^r(\mathbb{P}_\Delta, \mathcal{K}^*) \cong \; ''E_1^{r,0} \cong \mathbb{C}, \; and$$

$$\mathbb{H}^{r+l}(\mathbb{P}_\Delta, \mathcal{K}^*) = 0 \; for \; 1 \leq l \leq n - r - 1.$$

The result follows. □

It now follows immediately from the proposition that V is a Calabi-Yau variety. On the other hand, $\widehat{V}$ is also a complete intersection of global sections of semi-ample invertible sheaves $\widehat{\mathcal{L}}_i$ on X_Σ, the supporting polytopes of which are again $\Delta_1, \ldots, \Delta_r$. Hence $\widehat{V}$ still satisfy the hypotheses made in the above proposition, and so $\widehat{V}$ is a Calabi-Yau orbifold (again with at most terminal singularities).

When the E_j's are taken to be ample divisors of $\mathbb{P}_\Delta$, explicit formulae for the Hodge numbers $h^{n-r-1,1}(\widehat{V})$ and $h^{1,1}(\widehat{V})$ can be given in terms of the combinatorial data of Δ, as in the case of hypersurfaces:

$$\begin{aligned} h^{n-r-1,1}(\widehat{V}) &= \sum_{i=1}^{r}\Big(\sum_{J\subset I}(-1)^{r-|J|}l^*(\Delta_i + \sum_{j\in J}\Delta_j)\Big) - n - \\ &\quad \sum_{codim(\theta)=1}\Big(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\theta_j)\Big) \\ &\quad + \sum_{codim(\theta)=2} l^*(\theta^\circ)\Big(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\theta_j)\Big), \end{aligned}$$

$$\begin{aligned} h^{1,1}(\widehat{V}) &= \text{Card}\{\text{lattice points in faces of } \Delta^\circ \text{ of } dim \leq n-r-1\} - n \\ &\quad + \sum_{codim(\theta^\circ)=2} l^*(\theta)\Big(\sum_{J\subset I}(-1)^{r-|J|}l^*(\sum_{j\in J}\theta_j^\circ)\Big) \end{aligned}$$

where $I = \{1, \dots, r\}$, $J \subset I$ is a subset and $\theta_j \prec \Delta_j$ is the face corresponding to $\theta \prec \Delta$. We will not attempt to give a proof of these rather complicated formulae. Instead we remark that when $r = 1$, these formulae reduce to those in Section 2.3. More generally, Batyrev and Borisov defined the so-called *string-theoretic Hodge numbers* for V (cf. [3]). They are able to give explicit formulae for these numbers, which generalize all the formulae we present here.

As in the case of hypersurfaces, generic complete intersections in smooth toric Fano varieties give examples of smooth Calabi-Yaus. So we can regard the projective space $\mathbb{P}^n$ as a toric Fano variety. Denote by H a hyperplane. Then any effective divisor D on $\mathbb{P}^n$ is a multiple of H. In particular, the anticanonical divisor is given by $-K_{\mathbb{P}^n} = (n+1)H$. Let $E_i := d_i H$, $d_i \geq 2$ for $1 \leq i \leq r$ such that $d_1 + \ldots + d_r = n + 1$ (or equivalently $-K_{\mathbb{P}^n} = E_1 + \ldots + E_r$). This gives us a nef-partition of $-K_{\mathbb{P}^n}$. Let $f_i \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(E_i))$ be a generic section. Then f_i is a degree d_i homogeneous polynomial on $\mathbb{P}^n$. Let $V := \{f_1 = \ldots = f_r = 0\}$ be the complete intersection. Then V is a Calabi-Yau manifold of dimension $n - r$. The formulae in Section 2.4 certainly gives $h^{1,1}(V) = 1$, which also follows from the Lefschetz hyperplane theorem. The Euler number is given by

$$e(V) = \left[\frac{(n+1)n(n-1)}{6} - \frac{(n+1)^2 n}{2} + \frac{(n+1)^3}{3} - \frac{1}{3}\Big(\sum_{i=1}^{r} d_i^3\Big) \right] d_1 \ldots d_r.$$

In particular, when $r = n - 3$, we get smooth Calabi-Yau 3-folds. But $d_i \geq 2$, so

$$n + 1 = \sum_{i=1}^{n-3} d_i \geq 2(n-3) = 2n - 6$$

which implies that $n \leq 7$. The possible cases are well-known and are all listed as follows:

n	$d_i's$	description
4	$d = 5$	the well-known quintic threefold
5	$d_1 = d_2 = 3$	the intersection of two cubics
5	$d_1 = 2,\ d_2 = 4$	the intersection of a quadric and a quartic
6	$d_1 = d_2 = 2,\ d_3 = 3$	the intersection of two quadrics and one cubic
7	$d_1 = d_2 = d_3 = d_4 = 2$	the intersection of 4 quadrics

Similarly, regarding $X := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ as a toric Fano variety of dimension $n := n_1 + \ldots + n_k$, we can determine when can a complete intersection in X give us a Calabi-Yau manifold. Denote by H_j, $1 \leq j \leq k$, a hyperplane in $\mathbb{P}^{n_j}$ and let $I_i := \mathbb{P}^{n_1} \times \ldots \times H_i \times \ldots \times \mathbb{P}^{n_k}$. Then the effective divisors on X are generated by $I_1, \ldots, I_k$. The anticanonical divisor is given by $-K_X = (n_1+1)I_1 + \ldots + (n_k+1)I_k$. For $1 \leq i \leq r$, let $E_i := d_{i1}I_1 + \ldots + d_{ik}I_k$ with $d_{ij} > 0$ and $\sum_{j=1}^{k} d_{ij} \geq 2$; and satisfy:

$$\sum_{i=1}^{r} d_{ij} = n_j + 1 \ \text{ for } j = 1, \ldots, k.$$

This is a nef-partition of $-K_X$. Let $f_i \in H^0(X, \mathcal{O}_X(E_i))$ be a generic global section. Then f_i is a multi-homogeneous polynomial on X of multi-degree $(d_{i1}, \ldots, d_{ik})$. Let $V := \{f_1 = \ldots = f_k = 0\}$ be the complete intersection. Then V is an $(n-r)$-dimensional Calabi-Yau manifold. Again, the Hodge numbers can either be calculated by the formulae in Section 2.4 or by Lefschetz hyperplane theorem. As an example, take $X = \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then $-K_X = 4I_1 + 2I_2 + 2I_3$. Let $E_1 := 3I_1$ and $E_2 := I_1 + 2I_2 + 2I_3$. This gives a nef-partition and V is a complete intersection of the zero sets of polynomials of multi-degrees $(3, 0, 0)$ and $(1, 2, 2)$ on X. V is a 3-dimensional Calabi-Yau manifold with $e(V) = -48$, $h^{1,1}(V) = 9$ and $h^{2,1}(V) = 33$.

Before Batyrev's work, Calabi-Yau 3-folds have been constructed by physicists as hypersurfaces in 4-dimensional weighted projective spaces. However, to ensure that the hypersurface can be resolved to give a smooth Calabi-Yau, physicists have to impose the condition of *transversality* on weighted homogeneous polynomials, namely, the equations $f = 0$ and $df = 0$ have no common solutions. Finally physicists were able to list all the weighted $\mathbb{P}^4$ that admit a transverse polynomial. There are totally 7555 families. They then proceeded to plot the Hodge numbers $(h^{1,1}, h^{2,1})$ of these examples (these Hodge numbers are of course found by computer), and they found that the graph is *almost* symmetric. This suggest that most of the examples in the list are indeed mirror pairs of Calabi-Yau 3-folds. But a problem immediately arises due to the asymmetry: what are the missing mirrors? It turns out that these 'missing mirrors' can be found by using Batyrev's constructions. These mirrors are missing because they cannot be presented as transverse hypersurfaces in weighted $\mathbb{P}^4$. Nevertheless, using computer, it can be shown that the polytopes associated to them are reflexive and so fit to Batyrev's constructions. Now, the plot of Hodge numbers shows an exact, not just approximate, symmetry. For details, see [7].

Chapter 3

Calabi-Yau Manifolds by Quotients

In this chapter we construct Calabi-Yau 3-folds by taking quotients. Both free group actions and actions with fixed points will be considered.

3.1 Free Group Actions

First we recall the Bogomolov structure theorem (see [4]):

Theorem 3.1.1. *Let X be a compact Kähler manifold with $c_1(X) = 0$. Then there exists a finite unramified cover*

$$\widetilde{X} \to X$$

which is isomorphic to a product $\widetilde{X} \cong T \times \prod_i V_i \times \prod_j X_j$ where T is a complex torus; V_i is a simply connected projective manifold, of dimension ≥ 3, with trivial canonical bundle and $h^{p,0}(V_i) = 0$ for $0 < p < dim(V_i)$; and X_j is a simply connected holomorphic symplectic even dimensional Kähler manifold with trivial canonical bundle.

Note that the factors V_i's are simply connected Calabi-Yau manifolds, while the X_j's are the so-called *hyperkähler manifolds*. Now suppose that we have a compact Kähler 3-fold X with $c_1(X) = 0$ and $e(X) \neq 0$. Then $e(\widetilde{X}) \neq 0$ and hence no torus can appear in the above decomposition of $\tilde{X}$. There are also no factors of X_j's as they are even-dimensional. Therefore by the structure theorem, there exists a finite unramified cover

$$V \to X$$

where V is a simply connected Calabi-Yau 3-fold. This implies that for both V and X, we have:

$$h^{0,0} = h^{0,3} = 1, \; h^{0,1} = h^{0,2} = 0.$$

Hence X is also a Calabi-Yau 3-fold, with finite fundamental group $\pi_1(X)$. This suggests that we can start with a simply connected Calabi-Yau manifold V and construct new Calabi-Yau manifolds by finding groups, say G, acting freely on V. In this way, we may sometimes get Calabi-Yau manifolds with relatively small absolute value of the Euler number (since $\chi(V/G) = \chi(V)/|G|$), which is desirable for physicists studying Superstring Theory. We give some examples here.

Example 3.1.1 Consider the quintic V in $\mathbb{P}^4$ given by $\{f = 0\}$ where

$$f(z_0, z_1, z_2, z_3, z_4) := \sum_{i=0}^{4} z_i^5.$$

This is a simply connected Calabi-Yau 3-fold with Euler number -200. An $\mathbb{Z}_5 \times \mathbb{Z}_5$-action on V is given by the two generators:

$$\sigma : (z_0, z_1, z_2, z_3, z_4) \mapsto (z_5, z_1, z_2, z_3, z_4),$$

$$\tau : (z_0, z_1, z_2, z_3, z_4) \mapsto (\zeta z_1, \zeta^2 z_2, \zeta^3 z_3, \zeta^4 z_4, z_5),$$

where ζ is a primitive fifth root of unity. Note that a fixed point of σ must satisfy $(z_0, z_1, z_2, z_3, z_4) = \lambda(z_1, z_2, z_3, z_4, z_0)$ for some $\lambda \in \mathbb{C}$ and this implies $\lambda^5 = 1$. Hence no point of V is fixed under σ. On the other hand, the fixed points of τ in

$\mathbb{P}^4$ are $(1,0,0,0,0),(0,1,0,0,0),\ldots,(0,0,0,0,1)$, none of which is in V. Therefore, this is a free action. Let $X := V/(\mathbb{Z}_5 \times \mathbb{Z}_5)$. Then X is a Calabi-Yau manifold with fundamental group $\pi_1(X) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ and Euler number -8.

Example 3.1.2 Let $X := \mathbb{P}^3 \times \mathbb{P}^3$ with homogeneous coordinates $(x_0, x_1, x_2, x_3) \times (y_0, y_1, y_2, y_3)$. Using the notations in Example 2.5.2, we have $-K_X = 4I_1 + 4I_2$. Let $E_1 := 3I_1$, $E_2 := 3I_2$ and $E_3 := I_1 + I_2$. This gives a nef-partition of $-K_X$. Let V be the complete intersection of the zero sets of the polynomials

$$\sum_{i=0}^{3} x_i^3,\ \sum_{i=0}^{3} y_i^3,\ \sum_{i=0}^{3} x_i y_i.$$

Then V is a simply connected Calabi-Yau manifold with $e(V) = -18$, $h^{1,1}(V) = 14$ and $h^{2,1}(V) = 23$. Define $\sigma : X \to X$ by

$$\sigma : (x_0, x_1, x_2, x_3) \times (y_0, y_1, y_2, y_3) \mapsto (x_0, \omega^2 x_1, \omega x_2, \omega x_3) \times (y_0, \omega y_1, \omega^2 y_2, \omega^2 y_3),$$

where ω is a cubic root of unity. Then the restriction of σ (also denoted by σ) acts on V freely and $\sigma^3 = 1$. Let $G := \{1, \sigma, \sigma^2\} \cong \mathbb{Z}_3$ and $X := V/G$. Then X is a Calabi-Yau manifold with $\pi_1(X) = G \cong \mathbb{Z}_3$ and $e(X) = -6$. This is the famous *Tian-Yau manifold*, which was constructed by G. Tian and S.-T. Yau in the appendix of [36]. We can calculate $h^{2,1}(X)$ as follows. First note that since V is a Calabi-Yau, we have the isomorphism $H^1(V, T_V) \cong H^1(V, \Omega_V^2)$ where $H^1(V, T_V)$ is the space of infinitestimal deformations of V. By the deformation theory of Kodaira and Spencer, the linearly independent deformations can be represented by linearly independent homogeneous monomials. Now, the most general homogeneous cubic polynomial in $\mathbb{P}^3$ is given by $\sum a_{ijk} x_i x_j x_k$, providing 19 parameters (a_{ijk}), but only 4 of them are effective since the other 15 can be removed by a projective linear group. Hence a general cubic in $\mathbb{P}^3$ may be represented as:

$$\sum_{i=0}^{3} x_i^3 + a x_0 x_1 x_2 + b x_1 x_2 x_3 + c x_2 x_3 x_0 + d x_3 x_0 x_1$$

and the linearly independent deformations can be represented by the monomials $x_0x_1x_2$, $x_1x_2x_3$, $x_2x_3x_0$ and $x_3x_0x_1$. For X, we can, by making use of the PGL freedom on both $\mathbb{P}^3$, put the two cubics in the above standard form, giving us $4+4=8$ deformations. Then there will be no restriction on the defining equation of the hyperplane, so we have to count all terms x_iy_j, giving us $16-1=15$ deformations (one of them is removed by scaling a nonvanishing term to 1). Totally, there are $8+15=23=h^{2,1}(V)$ linearly independent deformations, represented by the monomials:

$x_ix_jx_k$	$y_iy_jy_k$	x_iy_j
$i,j,k \in \{0,1,2,3\}$	$i,j,k \in \{0,1,2,3\}$	except $i=j=0$
i,j,k all distinct	i,j,k all distinct	

It is easy to see that, out of these 23 terms, only 9 are invariant under G. Hence we have

$$h^{2,1}(X) = dim(H^1(V,\Omega_V^2)^G) = 9.$$

And so $h^{1,1}(X) = 6$.

3.2 Crepant Resolutions of Orbifolds

Generally speaking, it is more difficult to find free group actions than actions with non-trivial fixed points. In fact, non-free actions can also give us wonderful examples of Calabi-Yau manifolds, as in the case of Kummer surfaces. To deal with such examples, considerations of resolutions of singularities are necessary. We shall restrict ourselves to resolutions of *orbifolds*, i.e. varieties with only quotient singularities. This is because resolutions of singularities are most probable in this case, and more importantly, these varieties can indeed give us new examples of Calabi-Yau manifolds. We begin with resolving the singularities of orbifolds locally, namely, resolving quotient singularities.

Let n be a positive integer and G a non-trivial finite subgroup of $GL(n, \mathbb{C})$. Then G acts on $\mathbb{C}^n$ and gives the quotient singularity $\mathbb{C}^n/G$. As our purpose is to construct Calabi-Yau manifolds, we are interested in *crepant* resolutions of $\mathbb{C}^n/G$, i.e. resolutions that preserve the canonical bundle. However $g \in G$ acts on $\Lambda^{n,0}\mathbb{C}^n$ by $det(g)$, so that the canonical bundle of $\mathbb{C}^n/G$ is well-defined at 0 only if $det(g) = 1$ for all $g \in G$, i.e. $G \subset SL(n, \mathbb{C})$. It follows that $\mathbb{C}^n/G$ can have a crepant resolution only if $G \subset SL(n, \mathbb{C})$. From now on we shall assume that this condition holds unless stated otherwise. We will also assume that G is *abelian*, in which case toric methods are most efficient.

We follow the treatment given by Roan in [30], with some modifications. Since G is abelian, we may regard it as a diagonal subgroup of $SL(n, \mathbb{C})$:

$$G \subset \left\{ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \in (\mathbb{C}^*)^n : \prod_{i=1}^{n} \lambda_i = 1 \right\}.$$

Let $\mathbb{R}^n$ be the vector space consisting of all $n \times 1$ column vectors and $\{e_1, \ldots, e_n\}$ its standard basis. Define

$$exp : \mathbb{R}^n \to (\mathbb{C}^*)^n, \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} e^{2\pi\sqrt{-1}x_1} \\ \vdots \\ e^{2\pi\sqrt{-1}x_n} \end{pmatrix},$$

Let $N := exp^{-1}(G) \subset \mathbb{R}^n$. Then N is a free $\mathbb{Z}$-module of rank n and we can identify $N_{\mathbb{R}}$ with $\mathbb{R}^n$. Denote by $N' \subset N$ the standard lattice $\sum_{i=1}^{n} \mathbb{Z}e^i$, then as abelian groups, G is isomorphic to N/N'. Let σ be the cone in $N_{\mathbb{R}}$ generated by $\{e_1, \ldots, e_n\}$, i.e.

$$\sigma = \{x_1e_1 + \ldots x_ne_n : x_i \geq 0\}.$$

Then we have:

Proposition 3.2.1. *The affine variety U_σ is isomorphic to $\mathbb{C}^n/G$.*

Proof. In the appendix of [33], Roan and Yau gave an elementary proof of this. We give another proof here, following Fulton in [19]. Let $M \subset M'$ be the duals of $N' \subset N$. Also let σ' be the cone in $N'_{\mathbb{R}}$ corresponding to $\sigma \subset N_{\mathbb{R}}$, with $\mathbb{C}^n = U_{\sigma'} \to U_\sigma$. Now $G = N/N'$ acts on $\mathbb{C}[M']$ by

$$v \cdot \chi^{u'} = e^{2\pi\sqrt{-1}\langle u', v\rangle} \cdot \chi^{u'}$$

where $v \in N$ and $u' \in M'$. We claim that under this action, we have

$$\mathbb{C}[M']^G = \mathbb{C}[M].$$

To prove this, let $v_1, \ldots, v_n$ be a basis of N so that $k_1 v_1, \ldots, k_n v_n$ form a basis for N', for some positive integers k_i. Then $\mathbb{C}[M']$ is the Laurent polynomial ring in generators X_i, and $\mathbb{C}[M]$ is the Laurent polynomial ring in generators U_i, with $X_i = (U_i)^{k_i}$. An element $(a_1, \ldots, a_n) \in \bigoplus \mathbb{Z}/k_i\mathbb{Z} = G$ acts on monomials by multiplying $U_1^{l_1} \ldots U_n^{l_n}$ by $e^{2\pi\sqrt{-1}(\sum a_i l_i / k_i)}$, from which our claim follows. Intersecting $A_{\sigma'}$ with $\mathbb{C}[M']^G = \mathbb{C}[M]$, we get $A_{\sigma'}^G = A_\sigma$. Hence G acts on $U_{\sigma'}$ and

$$U_\sigma = U_{\sigma'}/G = \mathbb{C}^n/G.$$

□

Let $\Delta := \sigma \cap \{x = \sum_{i=1}^n x_i e_i \in N_{\mathbb{R}} : \sum_{i=1}^n x_i \leq 1\}$. Then Δ is a polytope in $N_{\mathbb{R}}$. Let $\widetilde{\Delta}$ be the subdivision of Δ such that:

(i) The vertices of $\widetilde{\Delta}$ are precisely the points in $\partial\Delta \cap N$; and

(ii) Any k-dimensional face of $\widetilde{\Delta}$ is the convex hull of $k+1$ lattice points in N.

Let Σ be the set of all cones over the faces of $\widetilde{\Delta}$. This gives us a partial resolution $\tau : X_\Sigma \to U_\sigma$, analogous to the one in Section 2.2. The following proposition then follows from the proofs of Propositions 2.2.2 and 2.2.3:

Proposition 3.2.2. *The map $\tau : X_\Sigma \to U_\sigma = \mathbb{C}^n/G$ is crepant; and the variety X_Σ has only terminal singularities.*

Remark 3.2.1. *For a toric variety X_Σ, the Euler number is equal to the number of n-dimensional cones in Σ. Hence in our case, $e(X_\Sigma) = |G|$.*

As mentioned in Section 2.2, orbifold terminal singularities are smooth in codimension 3, so in particular, we have:

Proposition 3.2.3. *For $n = 2,\ 3$, the resolved variety X_Σ is smooth.*

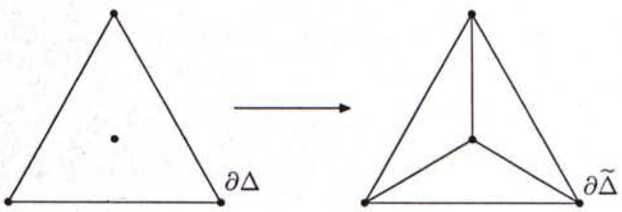
Remark 3.2.2. *For $n = 2$, there is only one choice of the subdivision $\widetilde{\Delta}$ and $\tau : X_\Sigma \to \mathbb{C}^2/G$ is the minimal resolution for the singularity of type A_k. However, there may be several choices for the subdivision in the case $n = 3$ and the resolved varieties X_Σ differ from each other by a flop.*

Let's look at some examples.

Example 3.2.1 Define $\Psi : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\Psi(z_1, z_2, z_3) = (\omega z_1, \omega z_2, \omega z_3)$$

where $\omega = e^{2\pi\sqrt{-1}/3}$, and let $G = \langle\Psi\rangle \cong \mathbb{Z}/3\mathbb{Z}$. In this case, the polytope Δ is the convex hull of $\{0, e_1, e_2, e_3\}$ and $\partial\Delta \cap N = \{0, e_1, e_2, e_3, \frac{1}{3}(e_1 + e_2 + e_3)\}$.



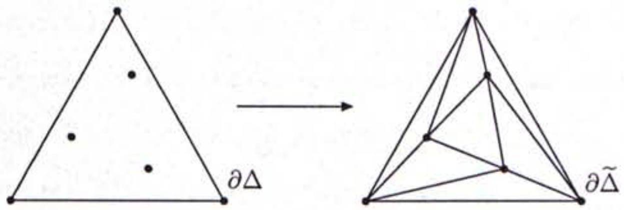
The resolved variety $\tau : X_\Sigma \to \mathbb{C}^3/G$ has Euler number $e(X_\Sigma) = |G| = 3$.

Example 3.2.2 Define $\Phi : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$\Phi(z_1, z_2, z_3) = (\zeta z_1, \zeta^2 z_2, \zeta^4 z_3)$$

where $\zeta = e^{2\pi\sqrt{-1}/7}$, and let $G = \langle\Phi\rangle \cong \mathbb{Z}/7\mathbb{Z}$. In this case, the polytope Δ is the convex hull of $\{0, e_1, e_2, e_3\}$ and

$$\partial\Delta \cap N = \{0, e_1, e_2, e_3, \frac{1}{7}e_1 + \frac{2}{7}e_2 + \frac{4}{7}e_3, \frac{2}{7}e_1 + \frac{4}{7}e_2 + \frac{1}{7}e_3, \frac{4}{7}e_1 + \frac{1}{7}e_2 + \frac{2}{7}e_3\}.$$



The resolved variety $\tau : X_\Sigma \to \mathbb{C}^3/G$ has Euler number $e(X_\Sigma) = |G| = 7$.

Now we have a resolution of 3-dimensional *abelian* quotient singularities by toric methods. The resolutions of general quotient singularities in dimension 3 is provided by Roan in [32]:

Theorem 3.2.1. *Let $G \subset SL(3, \mathbb{C})$ be any finite subgroup. Then there exists a smooth crepant resolution of quotient singularities*

$$\tau : X \to \mathbb{C}^3/G.$$

Roan's proof is by explicit constructions of the resolutions for all cases, making use of the classification of finite groups in $SL(3, \mathbb{C})$. The proof relies on earlier results by Ito [22], [23], Roan [31] and Markushevich [24]. Using this result, one can then prove the following theorem, also due to Roan [32]:

Theorem 3.2.2. *Let X be a 3-dimensional orbifold with only Gorenstein quotient singularities. Then X admits a crepant resolution.*

Proof. First we remark that 2-dimensional quotient singularities are classified and resolved classically. They are called the *Kleinian* or *Du Val* singularities. Each singularity $\mathbb{C}^2/G$ for a finite subgroup $G \subset SL(2, \mathbb{C})$ admits a unique crepant resolution. Now if X is a 3-dimensional orbifold with only Gorenstein quotient singularities, then its singularities consist of two types:

(i) curves locally having singularities of the form $(\mathbb{C}^2/G) \times \mathbb{C}$ where $G \subset SL(2, \mathbb{C})$; and

(ii) isolated points locally of the form $\mathbb{C}^3/G$ where $G \subset SL(3, \mathbb{C})$.

By the previous remark, singularities of type (i) admits a unique crepant resolution, while by Theorem 3.2.1, singularities of type (ii) each admits at least one crepant resolution. But since points of type (ii) are isolated, all these resolutions are compatible with each other. Glue them together then gives a crepant resolution of X. □

3.3 Examples From Complex Tori

By Theorem 3.2.2, one may now start with a 3-dimensional orbifold X with only Gorenstein quotient singularities and trivial canonical bundle, and consider a crepant resolution $\widetilde{X}$ of X. If it turns out that $e(\widetilde{X}) \neq 0$, then $\widetilde{X}$ is a smooth Calabi-Yau 3-fold by the discussion at the beginning of Section 3.1. In particular, we may choose X to be a finite quotient of a smooth manifold with trivial canonical bundle. This is in some sense a generalization of the Kummer constructions of $K3$ surfaces. Let's proceed with some examples.

Example 3.3.1 Consider the elliptic curve $E := \mathbb{C}/(\mathbb{Z} \oplus \omega\mathbb{Z})$ where $\omega = e^{2\pi\sqrt{-1}/3}$ is a primitive cubic root of unity. Set A_3 to be the triple product $E \times E \times E$. Let g_3 be the automorphism of A_3 by scalar multiplication by ω, and let $G := \{1, g_3, g_3^2\} \cong \mathbb{Z}/3\mathbb{Z}$. Take Y_3 to be the quotient A_3/G. This is a 3-dimensional orbifold with Gorenstein *cyclic* quotient singularities and trivial canonical bundle. Since E has 3 isolated singular points under the action of scalar multiplication by ω, Y_3 has 27 isolated singular points of type $\frac{1}{3}(1,1,1)$ which corresponds to Example 3.2.1. Now $\mathbb{C}^3/(\mathbb{Z}/3\mathbb{Z})$ has a *unique* toric resolution given by the blow-up of the singular point, so Y_3 also has a *unique* crepant resolution X_3 given by blowing-up the 27 singular points, replacing each by a copy of $\mathbb{P}^2$. The Euler number of X_3 is given by

$$e(X_3) = e(Y_3) - 27 + 27e(\mathbb{P}^2) = \frac{1}{3}(27 + 27) + 54 = 72.$$

Hence X_3 is a smooth Calabi-Yau 3-fold. We can also compute the Hodge numbers of X_3 as follows. A basis for $H^{2,1}(A_3)$ is given by $\{dz_i \wedge dz_j \wedge d\bar{z}_k : i,j,k \in \{1,2,3\}$ and $j < k\}$. The action of ω multiplies $dz_i \wedge dz_j \wedge d\bar{z}_k$ by $\omega^2\bar{\omega} = \omega$, so that $H^{2,1}(Y_3) = H^{2,1}(A_3)^G = 0$. Also, no new $(2,1)$-forms are introduced when Y_3 is resolved, which implies that $H^{2,1}(X_3) = H^{2,1}(Y_3) = 0$. Hence the Hodge numbers are given by: $h^{2,1}(X_3) = 0$ and $h^{1,1}(X_3) = 36$.

Example 3.3.2 Consider the *Klein quartic curve* $C = \{x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0\} \subset \mathbb{P}^2$, which is of genus $g = 3$. Let $A_7 := H^0(C, \Omega^1_C)^\vee / H_1(C, \mathbb{Z})$ be the Jacobian of C. This is an abelian 3-fold. Let g_7 be the automorphism of A_7 represented by the diagonal matrix $\mathrm{diag}(\zeta, \zeta^2, \zeta^4)$ where $\zeta = e^{2\pi\sqrt{-1}/7}$ is a primitive seventh root of unity. Denote by G the order 7 group generated by g_7 and take Y_7 to be the quotient A_7/G. Then again Y_7 is a 3-dimensional orbifold with Gorenstein *cyclic* quotient singularities and trivial canonical bundle. There are 7 singular points on Y_7 of type $\frac{1}{7}(1,2,4)$ which corresponds to Example 3.2.2. The crepant toric resolution X_7 of them is by replacing each with three $\mathbb{F}_2$'s crossing normally along the negative sections and fibers in a cyclic way. The Euler number of X_7 is given by

$$e(X_7) = e(Y_7) - 7 + 7 \cdot 7 = \frac{1}{7}(6 \cdot 7) + 42 = 48.$$

Hence X_7 is a smooth Calabi-Yau 3-fold. Similar to Example 3.3.1, all $(2,1)$-forms of A_7 are killed by the automorphism g_7, so we have

$$H^{2,1}(X_7) = H^{2,1}(Y_7) = H^{2,1}(A_7)^G = 0.$$

Hence the Hodge numbers are given by: $h^{2,1}(X_7) = 0$ and $h^{1,1}(X_7) = 24$.

It can be proved that the only smooth Calabi-Yau 3-folds given by the resolution of abelian 3-folds quotient by cyclic actions are the two given in the examples above [33], [30]. This indeed generalizes the Kummer constructions of $K3$ surfaces. However, in order to construct more examples, one has to consider quotients of abelian 3-folds by other(i.e. non-abelian) actions.

3.4 Complex Multiplication and Calabi-Yau Threefolds

The notion of complex multiplication originated with elliptic curves that have an *enhanced* endomorphism ring. Let $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ be an elliptic curve. Then E is said to have *complex multiplication* or simply be of *CM type* if the endomorphism ring $End(E)$ is seen to be larger than $\mathbb{Z}$. This is the case if and only if τ belongs to *a totally imaginary quadratic extension of* $\mathbb{Q}$, and the extension is then isomorphic to $End(E) \otimes_{\mathbb{Z}} \mathbb{Q}$.

This notion of complex multiplication can be generalized to higher dimensional varieties through the following (cf. [20]):

Definition 3.4.1. *Let V be a rational Hodge structure such that V is also a K-vector space for some CM-field K (i.e. a totally imaginary quadratic extension of a totally real number field), and such that the Hodge decomposition on V is stable under the action of K:*

$$xV^{p,q} \subset V^{p,q}, \qquad (x \in K, p, q \in \mathbb{Z}_{\geq 0}).$$

In particular, $K \hookrightarrow End_{Hod}(V)$. We will then say that V is a Hodge structure of CM-type (with field K).

Now an n-dimensional projective variety X is said to be of *CM-type* if the rational Hodge structure of weight n on $H^n(X, \mathbb{Q})$ is of CM-type.

Remark 3.4.1. *This definition of rational Hodge structures of CM-type is more general than the one which defines a rational Hodge structure of CM-type to be one with commutative Mumford-Tate group (cf. [5]).*

The main result of this section is to show that the two Calabi-Yau 3-folds constructed in the last section are of CM-type. The proof is due to Bert van Geemen [21].

Theorem 3.4.1. *The Calabi-Yau 3-folds X_3 and X_7 constructed in Section 3.3 are of CM-type.*

Proof. Let's consider X_3 first. It is obtained by taking quotient $Y_3 = E^3/\langle g_3 \rangle$, where E is the elliptic curve with $j(E) = 0$ and g_3 is an automorphism of order three of E^3. Next one takes X_3 to be a smooth crepant resolution of Y_3.

To see that X_3 is of CM type one relates its H^3 to the one of E^3. The most explicit way to do this is to take the fiber product Z of E^3 and X_3 over Y_3:

$$\begin{array}{ccc} Z & \longrightarrow & E^3 \\ \downarrow & & \downarrow \\ X_3 & \longrightarrow & Y_3 \end{array}$$

The variety Z is birational to E^3. The image of Z in the product of E^3 and X_3 is a codimension three subvariety (which may well be singular), and as such it has a cohomology class

$$[Z] \in H^6(E^3 \times X_3, \mathbb{Q}).$$

Using the Künneth decomposition of this cohomology group we get classes (component of $[Z]$):

$$[Z]_{a,6-a} \in H^a(E^3, \mathbb{Q}) \otimes H^{6-a}(X_3, \mathbb{Q}).$$

Poincaré duality on E^3 gives an isomorphism:

$$H^a(E^3, \mathbb{Q}) \cong H^{6-a}(E^3, \mathbb{Q})^* \qquad \text{(as dual vector space)}.$$

Finally there is the isomorphism for vector spaces V and W:

$$V^* \otimes W \cong \mathrm{Hom}(V, W).$$

Putting all these together the $[Z]_{a,6-a}$'s give homomorphisms which we denote by $[Z]_b$:

$$[Z]_b : H^b(E^3, \mathbb{Q}) \longrightarrow H^b(X_3, \mathbb{Q}).$$

Of course the situation is symmetric, so we also get homomorphisms which we denote by $[Z]_b^t$:

$$[Z]_b^t : H^b(X_3, \mathbb{Q}) \longrightarrow H^b(E^3, \mathbb{Q}).$$

Complexifying these $\mathbb{Q}$-vector spaces and extending $[Z]_b^t$ $\mathbb{C}$-linearly, the fact that the cohomology class of Z is of type (3,3) implies that the Hodge decomposition of these cohomology groups is preserved, so $[Z]_b^t$ induces $\mathbb{C}$-linear maps $[Z]_b^{t\,p,q}$:

$$[Z]_b^{t\,p,q} : H^{p,q}(X_3, \mathbb{Q}) \longrightarrow H^{p,q}(E^3, \mathbb{Q}).$$

Now the main point is that the image under $[Z]_3^{t\,3,0}$ of the holomorphic 3-form $\omega_3 \in H^{3,0}(X_3)$ is non-zero in $H^{3,0}(E^3)$. This follows essentially from the fact that pulling back ω_3 along $Z \to X_3$ gives an algebraic 3-form ω' on E^3. Up to some scalar, ω' is the image of ω_3 under $[Z]_3^{t\,3,0}$. One could also follow through the proof of the fact that X_3 is CY to see that ω_3 comes from an ω' on E^3 (which is the unique, up to scalar multiple, holomorphic three form on E^3 (note ω' is invariant under g_3)).

Using the automorphism of E^3 induced from the automorphism of order three on E and the Künneth decomposition of $H^3(E^3, \mathbb{Q})$ it is not hard to see that there is a two dimensional $\mathbb{Q}$-vector space V in $H^3(E^3, \mathbb{Q})$ such that $V \otimes_{\mathbb{Q}} \mathbb{C}$, a two dimensional complex subspace of $H^3(E^3, \mathbb{C})$, satisfies:

$$V \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}\omega' \oplus \mathbb{C}\overline{\omega'}.$$

The image of V under $[Z]_3$ in $H^3(X_3, \mathbb{Q})$ is then a two-dimensional $\mathbb{Q}$-vector space whose complexification contains ω_3 and hence:

$$([Z]_3 V) \otimes_{\mathbb{Q}} \mathbb{C} = H^{3,0}(X_3) \otimes H^{0,3}(X_3).$$

Thus to show that X_3 is of CM type is the same as showing that V is of CM-type.

Let's look more closely at the vector space V in $H^3(E^3, \mathbb{Q})$. Note first of all that since $V \otimes_{\mathbb{Q}} \mathbb{C}$ should contain $H^{3,0}$, which is contained in $H^{1,0} \otimes H^{0,1} \otimes H^{1,0}$, we should look for it in the summand $H = H^1(E,\mathbb{Q}) \otimes H^1(E,\mathbb{Q}) \otimes H^1(E,\mathbb{Q})$, this is an $2^3 = 8$ dimensional $\mathbb{Q}$ vector space. Let ϕ_i be the automorphism of order three of $H^1(E,\mathbb{Q})$ induced by ζ_3 on the i-th factor.

Note that

$$\phi_i\omega_3 = \zeta_3\omega_3 \qquad \text{for } i = 1, 2, 3.$$

Thus $\phi_2\phi_1^{-1}$ and $\phi_3\phi_1^{-1}$ act trivially on ω_3. Now let V be the subspace of H on which $\phi_2\phi_1^{-1}$ and $\phi_3\phi_1^{-1}$ act trivially. It is by definition a $\mathbb{Q}$ vector space and its complexification contains ω_3. To see it is two dimensional, we use the basis

$$dz_1 \wedge dz_2 \wedge dz_3, d\bar{z}_1 \wedge dz_2 \wedge dz_3, \ldots, dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$$

of $V \otimes \mathbb{C}$. Since ϕ_i acts by the cube root of unity ζ_3 on dz_i and is trivial on the other two dz_j, $d\bar{z}_j$ and acts by the cube root of unity $\bar{\zeta}_3$ on $d\bar{z}_i$, it is easy to check that only two of the 8 basis vectors are invariant, these are ω_3 and $\bar{\omega}_3$, which gives the result.

Now on V there still acts the automorphism induced by ϕ_1, it has order three and preserves the Hodge decomposition of V

$$V \otimes_{\mathbb{Q}} \mathbb{C} = V^{3,0} \oplus V^{0,3}, \qquad V^{3,0} = \mathbb{C}\omega_3.$$

So by definition, the V we just constructed does have CM.

For X_7, we have to take into account the fact that $H^1(C,\mathbb{Q})$, where C is the Klein curve, splits (in many ways) as the direct sum

$$H^1(C,\mathbb{Q}) \cong W^3, \qquad W \cong H^1(E',\mathbb{Q})$$

where $E' = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\sqrt{-7}$. The splitting can be seen by a result of Prapavessi [27] which says that $A_7 \cong E'^3$.

Tensor products of copies of W will split, as Hodge structures, into one and two dimensional simple Hodge substructures. The one dimensional ones are of type (p,p) of course, the two dimensional ones have CM by $\mathbb{Q}(\sqrt{-7})$.

In particular, there will be a two dimensional $\mathbb{Q}$-vector spaces W' in $H^3(E'^3, \mathbb{Q})$ such that $\omega_7' \in W' \otimes_{\mathbb{Q}} \mathbb{C}$, where ω_7' is the unique three form on C^3 invariant under g_7.

There will again be a correspondence between E'^3 and X_7, it will induce an isomorphism between W' and its image in $H^3(X_7, \mathbb{Q})$ and the complexification of the image of W' will contain ω_7. This implies that also X_7 has CM by $\mathbb{Q}(\sqrt{-7})$.

□

Chapter 4

Calabi-Yau Manifolds by Coverings

In this chapter we construct Calabi-Yau manifolds by cyclic coverings. The emphasis is on the construction of double octics, following Cynk [13], [11].

4.1 Cyclic Coverings

We begin with cyclic coverings of $\mathbb{P}^n$ branched along *smooth* divisors. Let $d \geq 2$ be an integer and let D be a smooth and reduced effective divisor in $\mathbb{P}^n$ with $\mathcal{O}_{\mathbb{P}^n}(D) = \mathcal{O}_{\mathbb{P}^n}(l)$ such that $d \big| l$. That is, there exist an integer $k \geq 1$ such that $l = dk$, so that

$$\mathcal{O}_{\mathbb{P}^n}(D) = \mathcal{O}_{\mathbb{P}^n}(dk) = \mathcal{O}_{\mathbb{P}^n}(k)^{\otimes d}.$$

Consider the d-cyclic covering $\pi : X \to \mathbb{P}^n$ branched along D. Then by standard theory of cyclic coverings (cf. [1], I.17), we have

$$K_X \cong \pi^*(K_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(k)^{\otimes(d-1)}) = \pi^*\mathcal{O}_{\mathbb{P}^n}(-(n+1) + k(d-1)),$$

which is trivial if and only if $k(d-1) = n+1$. We also have, in any case

$$\pi_*\mathcal{O}_X \cong \bigoplus_{j=0}^{d-1} \mathcal{O}_{\mathbb{P}^n}(k)^{-j}.$$

The fact that π is finite then implies

$$\begin{aligned} h^i(\mathcal{O}_X) &= h^i(\pi_*\mathcal{O}_X) \\ &= h^i(\mathcal{O}_{\mathbb{P}^n}) + h^i(\mathcal{O}_{\mathbb{P}^n}(-k)) + \ldots + h^i(\mathcal{O}_{\mathbb{P}^n}(-(d-1)k)) \\ &= 0 \end{aligned}$$

for $0 < i < n$. Hence if we are given integers $d \geq 2$ and $k \geq 1$ with $k(d-1) = n+1$, then the d-cyclic covering X of $\mathbb{P}^n$ branched along a reduced and smooth divisor D of degree $l = dk$ gives us a smooth Calabi-Yau n-fold. By the adjunction formula, the Euler number of X is given by

$$\begin{aligned} e(X) &= de(\mathbb{P}^n) - (d-1)e(D) \\ &= d(n+1) - (d-1)\sum_{i=0}^{n-1}(-1)^i \binom{n+1}{i} (dk)^{n-i}. \end{aligned}$$

For example, a double(respectively triple) cyclic cover of $\mathbb{P}^3$ branched along a smooth octic(respectively sextic) surface gives a smooth Calabi-Yau 3-fold with Euler number -296(respectively -204).

Remark 4.1.1. *The above constructions can be generalized to Fano manifolds. For the case of Fano threefolds, see [10].*

4.2 Admissible Blow-ups

In order to produce even more examples, we may introduce singularities in the branch locus and see if there are any suitable smooth models of the coverings. Our strategy is to consider an embedded resolution of the branch locus. However, to make sure that the canonical divisor does not change, we must be careful in choosing the types of blow-ups. This leads to the notion of *admissible blow-ups*, introduced by Cynk and Szemberg [12].

We restrict ourselves to dimension 3 and double covers, so let Y be a smooth 3-fold, $D \subset Y$ an even, reduced divisor and $Z \subset D$ a smooth irreducible proper subvariety. Consider the blow-up $\sigma : \widetilde{Y} \to Y$ with center Z and denote by E the exceptional divisor. Let $m_{Z/D}$ be the generic multiplicity of D at Z, and let $\widetilde{D}$ be the strict transform of D. Define the divisor $D^* \subset \widetilde{Y}$ by

$$D^* := \begin{cases} \widetilde{D} & \text{if } m_{Z/D} \text{ is even;} \\ \widetilde{D} + E & \text{if } m_{Z/D} \text{ is odd.} \end{cases}$$

Note that D^* is the only reduced and even divisor such that $\widetilde{D} \leq D^* \leq \sigma^*(D)$ where by $\sigma^*(D)$ we mean the total transform of D. The key definition is the following

Definition 4.2.1. *The blow-up $\sigma : \widetilde{Y} \to Y$ is called admissible (with respect to double covers) if*

$$K_{\widetilde{Y}} + \frac{1}{2}D^* \cong \sigma^*(K_Y + \frac{1}{2}D).$$

Now we determine all the admissible blow-ups:

Proposition 4.2.1. *On a smooth 3-fold Y, the only possible admissible blow-ups are as follows:*

1. *the blow-up of a curve Z with $m_{Z/D} = 2$ or 3; and*
2. *the blow-up of a point Z with $m_{Z/D} = 4$ or 5.*

Proof. Let r be the codimension of Z in Y. Define

$$\varepsilon(m_{Z/D}) = \varepsilon := \begin{cases} 0 & \text{if } m_{Z/D} \text{ is even;} \\ 1 & \text{if } m_{Z/D} \text{ is odd.} \end{cases}$$

Then we have

$$K_{\widetilde{Y}} \cong \sigma^* K_Y + (r-1)E, \; D^* = \sigma^* D - (m_{Z/D} - \varepsilon)E.$$

Hence

$$K_{\widetilde{Y}} + \frac{1}{2}D^* \cong \sigma^*(K_Y + \frac{1}{2}D) + (r - 1 - \frac{m_{Z/D} - \varepsilon}{2})E.$$

Thus σ is admissible if and only if $m_{Z/D} = 2(r-1) + \varepsilon$. The only possible cases are just as listed. □

4.3 Double Covers of $\mathbb{P}^3$ Branched Along Octic Arrangments

The result of Proposition 4.2.1 allows us to choose suitable branch loci. These are surfaces which locally look like an arrangement:

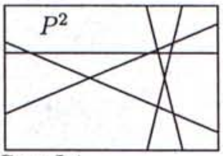
Definition 4.3.1. *Let $S_1, \ldots, S_r$ be smooth, irreducible surfaces in a smooth 3-fold U and let S be the sum of $S_1, \ldots, S_r$. Then S is called an arrangement if the followings are satisfied:*
1. For any $i \neq j$, S_i and S_j either intersect transversally along a smooth irreducible curve C_{ij} or they are disjoint; and
2. The curves C_{ij} and C_{kl} either coincide or they intersect transversally or they are disjoint.

We call an irreducible curve $C \subset S$ an *i-fold curve* if exactly i of the surfaces $S_1, \ldots, S_r$ pass through it; and a point $p \in S$ a *j-fold point* if exactly j of the surfaces $S_1, \ldots, S_r$ pass through it. In case $U = \mathbb{P}^3$ and $S_1, \ldots, S_r$ are surfaces of degree $d_1, \ldots, d_r$ respectively with $d_1 + \ldots + d_r = 8$, then S is called an *octic arrangement.* From now on, we shall focus on this case and construct Calabi-Yau 3-folds as double octics, i.e. double coverings of $\mathbb{P}^3$ branched along octic arrangements.

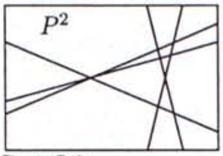
Theorem 4.3.1. *Let $S \subset \mathbb{P}^3$ be an octic arrangement with no q-fold curves for $q \geq 4$ and no p-fold points for $p \geq 6$. Then there exists a sequence $\sigma = \sigma_1 \circ \ldots \circ \sigma_s : \mathbb{P}^* \to \mathbb{P}^3$ of admissible blow-ups together with a smooth, even divisor $S^* \subset \mathbb{P}^*$ such that $\sigma(S^*) = S$ and the double covering X of $\mathbb{P}^*$ branched along S^* is a smooth Calabi-Yau 3-fold.*

Proof. We shall construct an explicit resolution of $S \subset \mathbb{P}^3$ through admissible blow-ups. This consists of 4 steps.

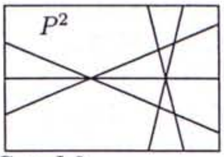
I. We first blow up all 5-fold points, $\sigma_1 : \mathbb{P}^3_{(1)} \to \mathbb{P}^3$. Let $p \in \mathbb{P}^3$ be a 5-fold point. Then the exceptional divisor E is isomorphic $\mathbb{P}^2$. Depending on the number of triple curves (i.e. 3-fold curves) on which p lies, we get one of the following configurations on E:



Case I-1



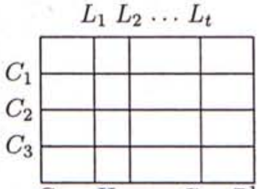
Case I-2



Case I-3

Denote by S_1 the sum of $\widetilde{S}$ and the exceptional divisors. Take S_1 to be the new branch locus. Note that S_1 contains no 5-fold points, but there are 5 new double lines and new 3-fold and 4-fold points.

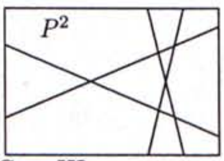
II. Secondly we blow up triple curves, $\sigma_2 : \mathbb{P}^3_{(2)} \to \mathbb{P}^3_{(1)}$. Let $C \subset S_1$ be a triple curve. The exceptional divisor is $E \cong C \times \mathbb{P}^1$. We have the following configuration on E:



Case II $C \times P^1$

Let S_2 be the new branch locus consisting of $\widetilde{S_1}$ plus the exceptional divisors. Now S_2 does not have any 5-fold points or triple curves, but we get new double lines and 4-fold points.

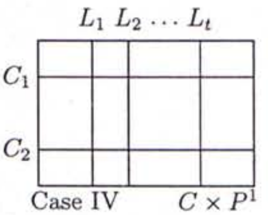
III. Next we blow up all 4-fold points in S_2, $\sigma_3 : \mathbb{P}^3_{(3)} \to \mathbb{P}^3_{(2)}$. Let $p \in S_2$ be a 4-fold point. The exceptional divisor is $E \cong \mathbb{P}^2$, and the configuration on E is as follows:



Case III

Let $S_3 := \widetilde{S}_2$ be the new branch locus. Then the only singularities of S_3 are double lines and triple points. Note that in this case, *no* new singularities are introduced.

IV. In the final step, we blow up all the double curves, $\sigma_4 : \mathbb{P}^* = \mathbb{P}^3_{(4)} \to \mathbb{P}^3_{(3)}$. Let $C \subset S_3$ be a double curve. The exceptional divisor is $E \cong C \times \mathbb{P}^1$, with the following configuration:



Take $S^* = S_4 := \widetilde{S}_3$. Then S^* is a smooth and even divisor of $\mathbb{P}^*$.

Let $\sigma := \sigma_4 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1 : \mathbb{P}^* \to \mathbb{P}^3$, and let $\pi : X \to \mathbb{P}^*$ be the double cover branching along S^*. Then since all the blow-ups σ_i, $i = 1, 2, 3, 4$, are admissible, we have, by the adjunction formula,

$$K_X \cong \pi^*(K_{\mathbb{P}^*} + \frac{1}{2}S^*) = \pi^*(\sigma^*(K_{\mathbb{P}^3} + \frac{1}{2}S)) = \mathcal{O}_X,$$

and by Serre duality,

$$\begin{aligned} h^1(\mathcal{O}_X) &= h^1(\mathcal{O}_{\mathbb{P}^*}) + h^1(\mathcal{O}_{\mathbb{P}^*}(-\frac{1}{2}S^*)) \\ &= h^1(\mathcal{O}_{\mathbb{P}^*}) + h^2(\mathcal{O}_{\mathbb{P}^*}(K_{\mathbb{P}^*} + \frac{1}{2}S^*)) \\ &= h^1(\mathcal{O}_{\mathbb{P}^3}) + h^2(\mathcal{O}_{\mathbb{P}^3}) \\ &= 0. \end{aligned}$$

Hence X is a smooth Calabi-Yau 3-fold. □

4.4 The Euler Number of X

First of all, we introduce some notations:

$e^*(S)$ = sum of Euler numbers of all components of S;

$E_i(S)$ = sum of Euler numbers of i-fold curves of S;

$p_j(S)$ = number of isolated j-fold points on S;

$p_j^k(S)$ = number of isolated j-fold points lying on exactly k triple curves.

We shall sometimes suppress the parameter S when there is no danger of confusion. Now we compute the changes of these data under an admissible blow-up.

Proposition 4.4.1. *Let S be an arrangement in a 3-fold U. Let $\sigma : V \to U$ be a blow-up of the type I, II, III or IV described in the proof of Theorem 4.3.1 with center Z, exceptional divisor E and $S^* = \widetilde{S} + \varepsilon E$. Then we have*

$$\begin{aligned} & 2e(U) - e^*(S) + 2E_2(S) - p_3(S) + 6E_3(S) + 12p_5^2(S) + 9p_5^1(S) + 6p_5^0(S) \\ = \; & 2e(V) - e^*(S^*) + 2E_2(S^*) - p_3(S^*) + 6E_3(S^*) + 12p_5^2(S^*) + 9p_5^1(S^*) + 6p_5^0(S^*) \end{aligned}$$

Proof. The proof is by explicitly verifying the above formula for each type of blow-up. In fact, we have the following table from [13]. Note that if Z is a q-fold line then t denotes the number of $(q+1)$-fold points on Z.

type	$e(V) - e(U)$	$e^*(S^*) - e^*(S)$	$E_2(S^*) - E_2(S)$	$E_3(S^*) - E_3(S)$
I-1	2	8	10	0
I-2	2	8	10	0
I-3	2	8	10	0
II	$e(Z)$	$2e(Z) + t$	$3e(Z) + 2t$	$-e(Z)$
III	2	4	0	0
IV	$e(Z)$	t	$-e(Z)$	0

type	$p_3(S^*) - p_3(S)$	$p_5^0(S^*) - p_5^0(S)$	$p_5^1(S^*) - p_5^1(S)$	$p_5^2(S^*) - p_5^2(S)$
I-1	10	-1	0	0
I-2	7	0	-1	0
I-3	4	0	0	-1
II	$3t$	0	0	0
III	0	0	0	0
IV	$-t$	0	0	0

□

Take $U = \mathbb{P}^3$ and apply the proposition to each blow-up in the sequence $\sigma = \sigma_1 \circ \sigma_2 \circ \sigma_3 \circ \sigma_4 : V := \mathbb{P}^* \to \mathbb{P}^3$. Note that S^* (in Theorem 4.3.1) is smooth, and for a double cover, $e(X) = 2e(V) - e^*(S^*)$. Hence we get the following corollary of the above proposition:

Corollary 4.4.1. *For the 3-fold X constructed in Theorem 4.3.1, we have*

$$e(X) = 8 - e^*(S) + 2E_2(S) - p_3(S) + 6E_3(S) + 12p_5^2(S) + 9p_5^1(S) + 6p_5^0(S).$$

We shall now compute the invariants used in the formula for $e(X)$.

Lemma 4.4.1. *For an octic arrangement $S \subset \mathbb{P}^3$, we have*

$$\begin{aligned} e^*(S) &= \sum_{i=1}^{r}(d_i^3 - 4d_i^2 + 6d_i); \text{ and} \\ 2E_2(S) + 6E_3(S) &= 2\sum_{i<j}(4 - d_i - d_j)d_i d_j. \end{aligned}$$

Proof. The first formula follows from the adjunction formula. For the second one, note that each triple curve is counted 3 times. Again by the adjunction formula, if C is a smooth complete intersection of surfaces of degree d_i and d_j, then $e(C) = (4 - d_i - d_j)d_i d_j$. This gives the second formula. □

Observe that since $\deg(S) = 8$, there are only two possibilities:

1. either there is 1 triple elliptic curve and no more triple curves, or
2. there are only triple lines.

Lemma 4.4.2. *For an octic arrangement in $\mathbb{P}^3$ with l_3 triple lines and no triple elliptic curves, we have*

$$\begin{aligned} p_3 + p_4 + 10p_5 - (p_4^1 + p_5^1 + 2p_5^2 - l_3) &= \sum_{i<j<k} d_i d_j d_k; \text{ and} \\ 5l_3 &= p_4^1 + 2p_5^2 + 4p_5^2. \end{aligned}$$

Proof. For the first formula, the right-hand-side is the number of triple points in case all intersections are transversal and reduced. Hence in case there are no triple lines, each triple point is counted by $\binom{3}{3} = 1$ time, each 4-fold point is counted by $\binom{4}{3} = 4$ times, and each 5-fold point is counted by $\binom{5}{3} = 10$ times. However, when triple lines do appear, any 4-fold point lying on such a line will only be counted by $4-1 = 3$ times (if counted properly). Similarly, a 5-fold point lying on exactly 1 triple line is counted by only $10 - 1 = 9$ times; and a 5-fold point lying on 2 triple lines will be counted by only $10 - 2 = 8$ times. One triple line corresponds to each success count. The first formula follows.

The second formula follows from the fact that the left-hand-side is the expected number of intersection points of 3 planes with the remaining quintic, while the right-hand-side is the number counted with multiplicities. □

By Corollary 4.4.1, Lemma 4.4.1 and Lemma 4.4.2, we finally get:

Theorem 4.4.1. *If $S \subset \mathbb{P}^3$ is an octic arrangement with no triple elliptic curves, then the Euler number of the Calabi-Yau 3-fold X constructed as a smooth model of the double cover of $\mathbb{P}^3$ branched along S is given by*

$$\begin{aligned} e(X) &= 8 - \sum_{i=1}^{r}(d_i^3 - 4d_i^2 + 6d_i) + 2\sum_{i<j}(4 - d_i - d_j)d_i d_j - \sum_{i<j<k} d_i d_j d_k \\ &\quad +4p_4^0 + 3p_4^1 + 16p_5^0 + 18p_5^1 + 20p_5^2 + l_3. \end{aligned}$$

Remark 4.4.1. *In [8], Cynk considered octic arrangements of surfaces with isolated singularities (of multiplicities 2, 4 and 5). Although this is just a slight generalization of the theory we presented here, more new examples of Calabi-Yau 3-folds can indeed be constructed. We quote the main result here without proof:*

Theorem 4.4.2 (Theorem 1.1 in [8]). *If $S \subset \mathbb{P}^3$ is an octic arrangement which contains isolated q-fold points for $q = 2, 4, 5$, then the double cover of $\mathbb{P}^3$ branched along S has a smooth model $\widehat{X}$ which is a Calabi-Yau 3-fold. Moreover, if S contains no triple elliptic curves, then*

$$\begin{aligned} e(\widehat{X}) &= 8 - \sum_{i=1}^{r}(d_i^3 - 4d_i^2 + 6d_i) + 2\sum_{i<j}(4 - d_i - d_j)d_id_j - \sum_{i<j<k} d_id_jd_k \\ &\quad +4p_4^0 + 3p_4^1 + 16p_5^0 + 18p_5^1 + 20p_5^2 + l_3 + 2m_2 + 36m_4 + 56m_5 \end{aligned}$$

where m_q is the number of isolated q-fold points for $q = 2, 4, 5$.

The resolution of 4-fold and 5-fold isolated points are exactly the same as in the proof of Theorem 4.3.1. The main difficulty lies in the resolution of 2-fold isolated points, or nodes, which requires the use of so-called small resolution.

4.5 The Hodge Numbers of X

To compute the Hodge numbers of the Calabi-Yau 3-fold X, we follow the approach of Cynk and van Straten [12]. As mentioned before, for a Calabi-Yau 3-fold X, $H^{2,1}(X) = H^1(\Omega_X^2) \cong H^1(T_X)$, is the space of infinitesimal deformations of X. Hence we may compute $h^{2,1}(X)$ by studying the latter. We start with the following lemma:

Lemma 4.5.1. *Let $\pi : X \to Y$ be a double cover of a smooth algebraic variety branched along a smooth divisor $D \subset Y$ with $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$ for some line bundle $\mathcal{L}$ on Y. Then*

$$H^1(T_X) \cong H^1(T_Y(\log D)) \oplus H^1(T_Y \otimes \mathcal{L}^{-1})$$

where $T_Y(\log D)$ is the logarithmic tangent sheaf, which is defined to be the subsheaf of the tangent sheaf T_Y consisting of derivations of $\mathcal{O}_Y$ which sends the ideal sheaf of D in Y into itself.

Proof. By the general theory of cyclic coverings [18], we have

$$\pi_* T_X = T_Y(\log D) \oplus (T_Y \otimes \mathcal{L}^{-1}).$$

Since π is finite,

$$H^1(T_X) \cong H^1(\pi_* T_X) \cong H^1(T_Y(\log D)) \oplus H^1(T_Y \otimes \mathcal{L}^{-1}).$$

□

For our purpose, take $Y = \mathbb{P}^*$ and $D = S^* \subset \mathbb{P}^*$, and choose a line bundle $\mathcal{L}^*$ on $\mathbb{P}^*$ such that $\mathcal{L}^{*\otimes 2} = \mathcal{O}_{\mathbb{P}^*}(S^*)$. Then the double cover $\pi : X \to \mathbb{P}^*$ branched along S^* gives

$$H^{2,1}(X) = H^1(\Omega^2_X) \cong H^1(T_{\mathbb{P}^*}(\log S^*)) \oplus H^1(T_{\mathbb{P}^*} \otimes \mathcal{L}^{*^{-1}}).$$

The problem then reduces to determining the two terms on the right-hand-side. We shall first deal with $H^1(T_{\mathbb{P}^*} \otimes \mathcal{L}^{*^{-1}})$.

Consider a blow-up $\sigma : \widetilde{Y} \to Y$ along a smooth subvariety $Z \subset Y$. Denote by E the exceptional divisor of σ and let $m \in \mathbb{Z}$ be such that $D^* = \sigma^* D - mE$, where D is an even, reduced divisor in Y, and D^* is the only even and reduced divisor in $\widetilde{Y}$ such that $\widetilde{D} \leq D^* \leq \sigma^* D$. Actually $m = 2\lfloor \frac{m_{Z/D}}{2} \rfloor$, so it is even. Define

$$\widetilde{\mathcal{L}} := \sigma^* \mathcal{L} \otimes \mathcal{O}_{\widetilde{Y}}(-\frac{m}{2}E)$$

and let $k := codim_Y(Z) - \frac{m}{2} - 1$. Then we have the following lemma:

Lemma 4.5.2. *Suppose $k = 0$, then*

$$H^1(T_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}}^{-1}) = \begin{cases} H^1(T_Y \otimes \mathcal{L}^{-1}) & \text{if } codim_Y(Z) < 2, \\ H^1(T_Y \otimes \mathcal{L}^{-1}) \oplus H^0(\det \mathcal{N}_{Z/Y} \otimes \mathcal{L}^{-1}) & \text{if } codim_Y(Z) = 2, \end{cases}$$

where $\mathcal{N}_{Z/Y}$ is the normal bundle of Z in Y.

Proof. We have the exact sequence

$$0 \to \sigma^*(\Omega^1_Y \otimes \mathcal{L} \otimes K_Y) \to \Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}} \to \Omega^1_{E/Z} \otimes \sigma^*(\mathcal{L} \otimes K_Y) \to 0.$$

As π is finite, we have

$$\sigma_*(\mathcal{O}_{\widetilde{Y}}) = \mathcal{O}_Y,\ R^i\sigma_*(\mathcal{O}_{\widetilde{Y}}) = 0, \text{ for } i \geq 1, \text{ and}$$

$$\sigma_*(\Omega^1_{E/Z}) = 0,\ R^1\sigma_*(\Omega^1_{E/Z}) \cong \mathcal{O}_Z,\ R^i\sigma_*(\Omega^1_{E/Z}) = 0 \text{ for } i \geq 2.$$

By the projection formula and the above exact sequence, we get

$$\begin{aligned} \sigma_*(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}) &\cong \Omega^1_Y \otimes \mathcal{L} \otimes K_Y, \\ R^1\sigma_*(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}) &\cong \mathcal{O}_Z \otimes \mathcal{L} \otimes K_Y, \\ R^i\sigma_*(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}) &= 0 \text{ for } i \geq 2. \end{aligned}$$

The Leray spectral sequence then gives

$$H^{n-1}(\Omega^1_{\widetilde{Y}} \otimes \widetilde{\mathcal{L}} \otimes K_{\widetilde{Y}}) \cong H^{n-1}(\Omega^1_Y \otimes \mathcal{L} \otimes K_Y) \oplus H^{n-2}(\mathcal{O}_Z \otimes \mathcal{L} \otimes K_Y).$$

The result now follows from Serre duality. □

By this lemma, Proposition 4.2.1 and the fact that $K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4) = \mathcal{L}^{-1}$, we conclude that

$$h^1(T_{\mathbb{P}^*} \otimes \mathcal{L}^{*^{-1}}) = \sum_C h^0(K_C) = \sum_C g_C$$

where the summation is over all blown up curves. If we further assume that S contains no triple elliptic curves, then a straightforward counting gives

$$h^1(T_{\mathbb{P}^*} \otimes \mathcal{L}^{*^{-1}}) = \binom{r}{2} - \frac{1}{2}\sum_{i<j}(4 - d_i - d_j)d_i d_j.$$

Next we shall determine $H^1(T_{\mathbb{P}^*}(\log S^*))$. By Theorem 4.1 of [12], $H^1(T_{\mathbb{P}^*}(\log S^*))$ is isomorphic to the space of *equisingular deformations* of $S \subset \mathbb{P}^3$, which is defined as follows:

Definition 4.5.1. *Given a pair $D \subset Y$ where Y is a smooth algebraic variety and D is a divisor in Y. Let $\sigma : \widetilde{Y} \to Y$ be a sequence $\sigma = \sigma_{n-1} \circ \ldots \circ \sigma_0$ where each $\sigma_i : Y_{i+1} \to Y_i$ is a blow-up of a smooth subvariety $Z_i \subset D_i^* \subset Y_i$, such that $D^* = D_n^*$ is smooth, $D_0^* = D$, $Y_0 = Y$ and $Y_n = \widetilde{Y}$. Let m_i be an integer such that $D_{i+1}^* = \sigma_i^* D_i^* - m_i E_i$ where $E_i \subset Y_{i+1}$ is the exceptional divisor of σ_i. Then an equisingular deformation of $D \subset Y$ is a simultaneous deformation of $D \subset Y$, which has simultaneous resolution, i.e. which can be lifted to a deformation of $Z_i \subset D_i^* \subset Y_i$ in such a way that the multiplicity of the deformation of D_i^* along the deformation of Z_i is at least m_i.*

In our case, $Y = \mathbb{P}^3$, $\widetilde{Y} = \mathbb{P}^*$ and $D = S \subset \mathbb{P}^3$. One of the key ideas of [12] is to describe the space of equisingular deformations of $S \subset \mathbb{P}^3$ (which is quite abstract) in a very concrete way. We shall recall the result here without proof. Denote by J_f the *Jacobian ideal* of S:

$$J_f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_3}\right)$$

where f is a homogeneous equation for S. Let Z_i, $i = 0, \ldots, n-1$ be the multiple points and curves of the octic arrangement S, and m_i the corresponding multiplicities. Denote by $I(Z_i)$ the homogeneous ideal of Z_i.

Definition 4.5.2. *The equisingular ideal of $S \subset \mathbb{P}^3$ is defined as*

$$I_{eq}(S) = \bigcap_{i=0}^{n-1} \left(I(Z_i)^{m_i} + J_f\right).$$

Now Theorem 4.5 of [12] states that

Theorem 4.5.1. *The space of equisingular deformations of $S \subset \mathbb{P}^3$ is isomorphic to the space of degree 8 forms in the quotient of the equisingular ideal modulo the Jacobian ideal. Hence we have*

$$H^1(T_{\mathbb{P}^*}(\log S^*)) \cong \left(I_{eq}(S)/J_f\right)_8.$$

That is, $h^1(T_{\mathbb{P}^*}(\log\, S^*)) = dim_{\mathbb{C}}\big(I_{eq}(S)/J_f\big)_8$. And altogether, we have the following formula for the Hodge number $h^{2,1}(X)$:

$$h^{2,1}(X) = dim_{\mathbb{C}}\big(I_{eq}(S)/J_f\big)_8 + \sum_C g_C$$

and if S contains no triple elliptic curves:

$$h^{2,1}(X) = dim_{\mathbb{C}}\big(I_{eq}(S)/J_f\big)_8 + \binom{r}{2} - \frac{1}{2}\sum_{i<j}(4 - d_i - d_j)d_i d_j$$

Remark 4.5.1. *(i) The first term in the formula, i.e. the dimension of degree 8 forms in the quotient of the equisingular ideal modulo the Jacobian ideal, can be computed using a computer algebra system, as in [11].*

(ii) Since X is a Calabi-Yau 3-fold and we have determined both $e(X)$ and $h^{2,1}(X)$, we can also compute $h^{1,1}(X) = \frac{1}{2}e(X) + h^{2,1}(X)$. In this way, one may compile a large list of Calabi-Yau 3-folds together with their invariants by using computer. In [11], Cynk and Meyer constructed such a list of examples that correspond to arrangements of eight planes defined over $\mathbb{Q}$. Note that for these examples,

$$h^{2,1}(X) = dim_{\mathbb{C}}\big(I_{eq}(S)/J_f\big)_8,$$

i.e. all deformations of X come from equisingular deformations of S in $\mathbb{P}^3$. They produce seven rigid Calabi-Yau 3-folds (i.e. $h^{2,1} = 0$) and 14 examples with $h^{2,1} = 1$, and with equations given (cf. [11]).

4.6 $K3$-Fibrations and Modularity

Let X be a Calabi-Yau 3-fold which admits a K3-fibration, i.e. a proper and surjective morphism

$$\Phi : X \to \mathbb{P}^1$$

such that the general fibres are $K3$ surfaces. For any fibre F, we denote by $l(F)$ the number of irreducible components of F; and we also let ρ be the Picard rank of a general fibre. Then we have the following formula:

Theorem 4.6.1.

$$h^{1,1}(X) = \rho + 1 + \sum_{\textit{reducible } F} (l(F) - 1)$$

where the summation is over all reducible fibres of Φ.

Sketch of Proof. Let $S \subset \mathbb{P}^1$ be the degeneration locus of Φ, $Y_0 = \mathbb{P}^1 \backslash S$ and $X_0 = \Phi^{-1}(Y_0)$. Let $F = \Phi^{-1}(t)$ be a general fibre of Φ. By the results of Deligne in [16], we have the following maps

$$H^2(X, \mathbb{Q}) \xrightarrow{a} H^2(X_0, \mathbb{Q}) \xrightarrow{b} H^0(Y_0, R^2\Phi_*\mathbb{Q}) \xrightarrow{c} H^2(F, \mathbb{Q})$$

where $b \circ a$ is surjective and c is injective. The composite

$$r := c \circ b \circ a : H^2(X, \mathbb{Q}) \to H^2(F, \mathbb{Q})$$

is the so-called *restriction map*. Deligne's results tell us that r is a morphism of Hodge structures. But X is a Calabi-Yau 3-fold, we have $H^{2,0}(X) = H^{0,2}(X) = 0$ and so

$$H^2(X, \mathbb{Q}) = H^2(X, \mathbb{Q}) \cap H^{1,1}(X) = NS(X)_{\mathbb{Q}}$$

where the last one is the Néron-Severi group of X. Hence we can regard r as the map

$$r : NS(X)_{\mathbb{Q}} \to NS(F)_{\mathbb{Q}} = H^2(F, \mathbb{Q}) \cap H^{1,1}(F)$$

defined over $\mathbb{Q}$. Note that the kernel of r consists of the components in the fibres of Φ. We thus arrive at the inequality

$$h^{1,1}(X) = \rho(X) \leq \rho(F) + 1 + \sum_{\text{reducible } F} (l(F) - 1).$$

Now we look more closely at the map $c : H^0(Y_0, R^2\Phi_*\mathbb{Q}) \to H^2(F, \mathbb{Q})$. Still by Deligne's results, we know that actually $H^0(Y_0, R^2\Phi_*\mathbb{Q})$ is isomorphic to the part of $NS(F)_{\mathbb{Q}} = H^2(F, \mathbb{Q}) \cap H^{1,1}(F)$ invariant under monodromy. Hence it suffices to show that all algebraic cycles on the general fibre F are invariant under monodromy. We shall see that this is the case for any (nonconstant) pencil of $K3$ surfaces.

So suppose that $\Phi : X \to C$ is a pencil of $K3$ surfaces. Let F be a general fibre. Assume that F has degree $2n$ and let $NS(F) \hookrightarrow H^2(F, \mathbb{Z}) \cong L = U^{\oplus 3} \oplus E_8^{\oplus 2}$ denotes the Néron-Severi lattice of F. There exists a coarse moduli space $\mathcal{M}_{2n}$ for degree $2n$ polarized $K3$ surfaces, equipped (may be after base change) with a universal family $\mathcal{X}_{2n} \to \mathcal{M}_{2n}$. Hence we have a proper morphism

$$\psi : C \to \mathcal{M}_{2n}$$

such that $\Phi : X \to C$ is the pull-back family:

$$\begin{array}{ccc} X = C \times_{\mathcal{M}_{2n}} \mathcal{X}_{2n} & \longrightarrow & \mathcal{X}_{2n} \\ {\scriptstyle \Phi}\downarrow & & \downarrow \\ C & \xrightarrow{\psi} & \mathcal{M}_{2n} \end{array}$$

But at the same time, we have the coarse moduli space $\mathcal{M}_M$ for M-polarized $K3$ surfaces for a primitive embedding of lattices $M \hookrightarrow L$ (cf. [17]). This is a subvariety (or submoduli) in the moduli space $\mathcal{M}_{2n}$. For a fixed rank, say ρ, there are at most countably many subvarieties of $\mathcal{M}_{2n}$, each of which is the coarse moduli for M-polarized $K3$ surfaces for some $M \hookrightarrow L$ with rk $M = \rho$. Now the image of C under ψ is an algebraic curve in $\mathcal{M}_{2n}$. Since there are just countably many such subvarieties, there must be one on which $\psi(C)$ have intersections at uncountably many points. This is possible only if the whole image $\psi(C)$ lies in that subvariety (or submoduli). Hence we can conclude that the family $\Phi : X \to C$ is the pull-back of the universal family associated to the coarse moduli space for $NS(F)$-polarized $K3$ surfaces. However, by construction, the algebraic cycles of a general fibre in the universal family are all invariant under monodromy. The result follows. □

Remark 4.6.1. *(i) The idea of the proof is suggested by Professor Kang Zuo.*

(ii) *One can of course consider a pencil of Calabi-Yau n-folds such that the total space is a Calabi-Yau $(n+1)$-fold. Unfortunately, the proof we give here relies essentially on the properties of the moduli spaces of $K3$ surfaces. The author does not know whether it can be generalized to higher dimensions.*

Consider the case when X is a double octic. Fix a double line, i.e. a transversal intersection of two planes, in the arrangement. (Since it is an arrangement of eight planes, double lines always exist. In fact, a generic arrangement of eight planes has 28 double lines and 56 triple points as singularities.) Recall that in the embedded resolution of the branch locus, each double line has to be blown up. The blow-up of the fixed double line then induces a $K3$-fibration on X:

$$\Phi : X \to \mathbb{P}^1.$$

The fibration structure can also be seen as follows. Consider a plane (i.e. a $\mathbb{P}^2$) in $\mathbb{P}^3$ passing through the fixed double line. This plane will intersect with the six remaining planes, i.e. those on which the double line does not lie, in six lines. These can be considered as an arrangement of six lines on $\mathbb{P}^2$, which will generally give rise to a $K3$ surface, just like an octic arrangement generally gives us a Calabi-Yau 3-fold. Turning around the plane along the fixed double line, so that it cuts the six planes in different directions, then gives us a pencil of $K3$ surfaces. In this way, it is easy to see that blowing up the double line gives us a $K3$-fibration on X. We should remark that the idea of this construction of $K3$-fibrations on X is also due to Professor Zuo.

Now we give some examples and use Theorem 4.6.1 to calculate the Picard rank of a general fibre. We will make use of the examples given in the table in [11].

Example 4.6.1 Consider Arrangement no.2 in [11]. The equation of the octic arrangement is given by

$$xyzt(x+y)(y+z)(z+t)(t+x) = 0$$

where $(x : y : z : t)$ denote the homogeneous coordinates on $\mathbb{P}^3$. The non-generic singularities (i.e. those other than triple points and double lines) are given as follows:

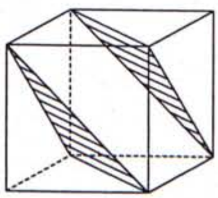
$$\begin{aligned}
p_5^2 \text{ points} &: (1:0:0:0),\ (0:1:0:0),\ (0:0:1:0),\ (0:0:0:1),\\
p_4^1 \text{ points} &: (1:-1:0:0),\ (0:1:-1:0),\ (0:0:1:-1),\ (1:0:0:-1),\\
p_4^0 \text{ points} &: (1:-1:1:-1),\\
\text{triple lines} &: x=y=0,\ y=z=0,\ z=t=0,\ t=x=0.
\end{aligned}$$

The Hodge numbers of X are $h^{1,1}(X) = 70$ and $h^{2,1}(X) = 0$, so that X is a rigid Calabi-Yau 3-fold. Fix the double line $x + y = y + z = 0$ and consider the $K3$ fibration induced by the blow-up of this line. One may check that there are 3 reducible fibres F_1, F_2 and F_3 in the fibration and $l(F_1) = l(F_2) = 21$, $l(F_3) = 11$. Hence the Picard rank of a general fibre in this fibration is given by

$$\rho = 70 - 1 - (20 + 20 + 10) = 19.$$

Example 4.6.2 Consider Arrangement no.85 in [11], with equation:

$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(x+y+z-t)(x+y+z+t) = 0.$$



There are only p_4^0 points other than generic singularities:

$$\begin{aligned}
p_4^0 \text{ points} : \ & (1:1:-1:-1),\ (1:-1:1:-1),\ (1:-1:-1:1),\\
& (1:0:0:0),\ (0:1:0:0),\ (0:0:1:0),\\
& (1:-1:0:0),\ (0:1:-1:0),\ (1:0:-1:0),\\
& (1:1:-1:1),\ (1:-1:1:1),\ (-1:1:1:1).
\end{aligned}$$

The Hodge numbers of X are $h^{1,1}(X) = 44$ and $h^{2,1}(X) = 0$, so X is also a rigid Calabi-Yau 3-fold. Fix the double line $y + t = z + t = 0$ and consider the $K3$ fibration induced by the blow-up of this line. In this case we have 4 reducible fibres F_1, F_2, F_3 and F_4 in the fibration and $l(F_1) = l(F_2) = 11$, $l(F_3) = l(F_4) = 3$. Hence the Picard rank of a general fibre in this fibration is given by

$$\rho = 44 - 1 - (10 + 10 + 2 + 2) = 19.$$

Example 4.6.3 Consider Arrangement no.1 in [11]. The equation is given by

$$xyzt(x + y)(y + z)(z + t)(Bt + Ax) = 0$$

where $(A : B)$ is a generic point in $\mathbb{P}^1$. The non-generic singularities are given as follows:

$$\begin{aligned} p_5^2 \text{ points} &: (1:0:0:0),\ (0:1:0:0),\ (0:0:1:0),\ (0:0:0:1), \\ p_4^1 \text{ points} &: (1:-1:0:0),\ (0:1:-1:0),\ (0:0:1:-1),\ (B:0:0:-A), \\ \text{triple lines} &: x = y = 0,\ y = z = 0,\ z = t = 0,\ t = x = 0. \end{aligned}$$

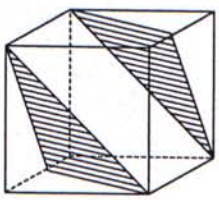
The Hodge numbers of X are $h^{1,1}(X) = 69$ and $h^{2,1}(X) = 1$. Fix the double line $x = y + z = 0$ and consider the $K3$ fibration induced by the blow-up of this line. There are 3 reducible fibres F_1, F_2 and F_3 in the fibration and $l(F_1) = 30$, $l(F_2) = 20$, $l(F_3) = 3$. Hence the Picard rank of a general fibre in this fibration is given by

$$\rho = 69 - 1 - (29 + 19 + 2) = 18.$$

Example 4.6.4 Consider Arrangement no.83 in [11], with equation:

$$(x-t)(x+t)(y-t)(y+t)(z-t)(z+t)(Ax+By+Bz-At)(Ax+By+Bz+At) = 0$$

and $(A : B)$ denotes a generic point in $\mathbb{P}^1$.



There are also only p_4^0 points other than generic singularities:

$$\begin{aligned} p_4^0 \text{ points } : \ & (1:1:-1:-1),\ (1:-1:1:-1), \\ & (1:0:0:0),\ (0:1:0:0),\ (0:0:1:0), \\ & (B:-A:0:0),\ (0:1:-1:0),\ (B:0:-A:0), \\ & (1:1:-1:1),\ (1:-1:1:1). \end{aligned}$$

The Hodge numbers of X are $h^{1,1}(X) = 41$ and $h^{2,1}(X) = 1$, so X is also a rigid Calabi-Yau 3-fold. Fix the double line $y + t = z + t = 0$ and consider the $K3$ fibration induced by the blow-up of this line. There are 4 reducible fibres F_1, F_2, F_3, F_4 and $l(F_1) = l(F_2) = 11$, $l(F_3) = l(F_4) = 2$. Hence the Picard rank of a general fibre in this fibration is given by

$$\rho = 41 - 1 - (10 + 10 + 1 + 1) = 18.$$

Remark 4.6.2. *Note that the rigid Calabi-Yau 3-folds in Example 4.6.1 and 4.6.2 are special fibres (when $A = B = 1$) in the one dimensional families in Example 4.6.3 and 4.6.4 respectively.*

In [35], Sun, Tan and Zuo considered Calabi-Yau 3-folds fibred by non-constant semi-stable $K3$ surfaces. They get the following result about modularity of Calabi-Yau 3-folds:

Theorem 4.6.2. *([35], Corollary 0.4) Let $f : X \to \mathbb{P}^1$ be a Calabi-Yau 3-fold fibred by non-constant semi-stable $K3$ surfaces. Then the following hold true:*

(i) If the iterated Kodaira-Spencer map of f is non-zero, then f has at least 4 singular fibres. If f has exactly 4 singular fibres, then X is rigid, the general fibres of f have Picard number 19 and X is birational to the Nikulin-Kummer construction of a square product of a family of elliptic curves $g : E \to \mathbb{P}^1$. After passing to (if necessary) a double cover $E' \to E$, the family $g' : E' \to \mathbb{P}^1$ is one of the 6 modular families of elliptic curves constructed by Beauville.

(ii) If the iterated Kodaira-Spencer map of f is zero, then f has at least 6 singular fibres. If f has exactly 6 singular fibres, then X is non-rigid, the general fibres of f have Picard number 18 and $\mathbb{P}^1 \backslash S \simeq \mathcal{H}/\Gamma$, where Γ is a subgroup of $SL(2, \mathbb{Z})$ of index 24.

This theorem shows that, among others, the modularity of some rigid Calabi-Yau 3-folds fibred by semi-stable $K3$ surfaces can *directly* be seen from their geometry. Unfortunately, the examples we give above do not fit in this theorem because those fibrations are *not* semi-stable, although it is reasonable to expect that semi-stability does not affect the modularity.

Nevertheless, Cynk and Meyer have checked (in [11]) that the 7 rigid Calabi-Yau 3-folds they constructed, including the two examples above, are indeed modular in another sense, which is described as follows. Let X be a Calabi-Yau 3-fold defined over $\mathbb{Q}$. Assume that X has a suitable integral model. The L-series of X is then defined to be the L-series of the Galois representation on $H^3_{et}(\bar{X}, \mathbb{Q}_l)$. That is,

$$L(X, s) := L(H^3_{et}(\bar{X}, \mathbb{Q}_l), s).$$

The modularity conjecture for rigid Calabi-Yau 3-folds is the following (cf. [34]):

Conjecture 4.6.1. *Any rigid Calabi-Yau 3-fold X defined over $\mathbb{Q}$ is modular in the sense that the L-series of X coincides with the Mellin transform of the L-series of a cusp form f of weight 4 on $\Gamma_0(N)$, where N is a positive integer divisible by the primes of bad reduction. More precisely, we have, up to a finite Euler factor,*

$$L(X, s) = L(f, s) \quad \text{for some} \quad f \in S_4(\Gamma_0(N)).$$

Cynk and Meyer verified this conjecture for the 7 rigid Calabi-Yau 3-folds they constructed. It is interesting to know how we can see the modularity directly from geometry, as in Theorem 4.6.2. In fact, we already have some information, at least for the examples we consider. For instance, for the two rigid Calabi-Yau

3-folds we consider, we have found $K3$-fibrations on them whose general fibres have Picard number 19. But the moduli space of M-polarized $K3$ surfaces with rk $M = 19$ is one-dimensional and is an arithmetic quotient of bounded symmetric domain. This suggests that the $K3$-fibrations, or the rigid Calabi-Yau 3-folds, are modular. On the other hand, for the Calabi-Yau 3-folds with $h^{2,1}(X) = 1$, we have found $K3$-fibrations whose general fibres have Picard number 18. We do expect that they are modular, but it remains to give a sounding geometric picture.

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