THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2070A Algebraic Structures 2019-20
Homework 1
Due Date: 12th September 2019

Compulsory Part

1. Let
$$T = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C},\ xy = 1 \right\}.$$
Show that T is a group under matrix multiplication.

2. Show that for all g in a group G, we have $(g^{-1})^{-1} = g$.

3. Consider the set
$$A := \{ f : \mathbb{R} \to \mathbb{R} : \text{ there exist } a, b \in \mathbb{R} \text{ with } a \neq 0 \text{ such that } f(x) = ax + b\ \forall\, x \in \mathbb{R} \}$$
of affine functions on $\mathbb{R}$. Show that A forms a group under composition of maps.

Optional Part

1. Determine whether the given set equipped with the given binary operation is a group (if it is, give a proof; if it is not, explain why):
   (a) The set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers, equipped with addition.
   (b) The set $\mathbb{R}_{>0}$ of positive real numbers, equipped with multiplication.
   (c) The set $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ of even integers, equipped with addition. (How about odd integers?)
   (d) The set $U := \{z \in \mathbb{C} : |z| = 1\}$ of complex numbers with modulus 1, equipped with multiplication.
   (e) The set $\{z \in \mathbb{C} : \operatorname{Im} z = 1\}$ of complex numbers with imaginary part equals to 1, equipped with multiplication.
   (f) The set $M_{m\times n}(\mathbb{C})$ of $m \times n$ complex matrices, equipped with matrix addition.
   (g) The set of 2×2 matrices with integer coefficients whose determinants are non-zero, equipped with matrix multiplication.
   (h) The set $H = \mathbb{R} \times \mathbb{R}$, equipped with the operation $*$ defined by
   $$(x_1, y_1) * (x_2, y_2) = (x_1 + x_2, y_1 + y_2 + x_1 x_2)$$
   for $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

2. Let
$$R = \{ r \in \mathbb{Q} : \text{ there exist } n \in \mathbb{Z}_{>0} \text{ such that } 2^n r \in \mathbb{Z} \}.$$
Is R a group under addition? Justify your answer.

3. Let G be a group. Show that the equation

$$x^2 = x$$

has a *unique* solution in G.

4. Let G_1, G_2 be groups. Show that the Cartesian product $G_1 \times G_2$ is a group under the operation

$$(a_1, b_1) * (a_2, b_2) := (a_1 *_1 a_2, b_1 *_2 b_2)$$

for $a_1, a_2 \in G_1$ and $b_1, b_2 \in G_2$, where $*_1, *_2$ are the group operations of G_1, G_2 respectively. The group $G_1 \times G_2$ is called the **direct product** of G_1 and G_2. Similarly, one can define the direct product of *any* number of groups.

5. The **quaternion group** is defined as follows:

$$Q = \{1, -1, i, j, k, -i, -j, -k\},$$

where the group operation is written multiplicatively, the symbol 1 denotes the identity element, and $-i, -j, -k$ denotes $(-1)i$, $(-1)j$, $(-1)k$, respectively.

Moreover, by definition -1 commutes with every element of the group (for instance, $(-1)i = i(-1) = -i$), and the symbols i, j, k satisfy the following relations:

$$(-1)^2 = 1, \quad i^2 = j^2 = k^2 = ijk = -1.$$

(a) Show that $ij = k$ and $jk = i$.

(b) Show that $ij = -ji$.